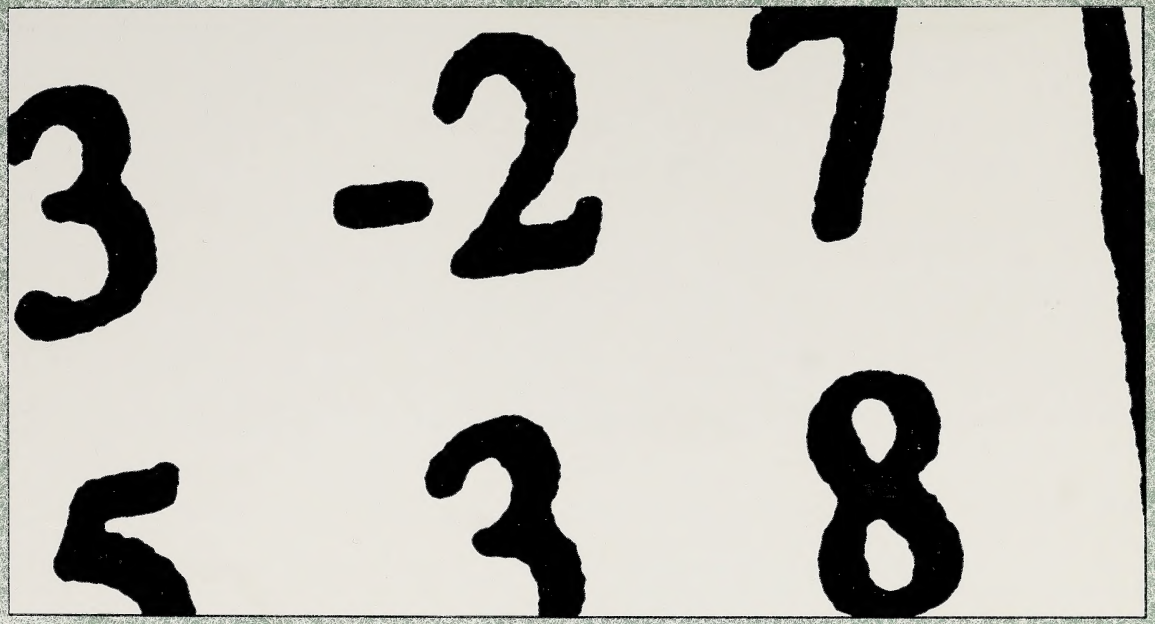
MATHEMATICS 3



Distance Learning

UNIT 8: SYSTEMS OF LINEAR EQUATIONS

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EDUCATION

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Welcome

Distance Learning

You have chosen an alternate form of learning that allows you to work at your own pace. You will be responsible for your own schedule, for disciplining yourself to study the units thoroughly, and for completing your units regularly. We wish you much success and enjoyment in your studies.

Mathematics 31 Student Module Unit 8 Systems of Linear Equations Alberta Distance Learning Centre ISBN No. 0-7741-0361-2

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General Information

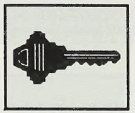
This information explains the basic layout of each booklet.

- **What You Already Know** and **Review** are to help you look back at what you have previously studied. The questions are to jog your memory and to prepare you for the learning that is going to happen in this unit.
- As you begin each **Topic**, spend a little time looking over the components. Doing this will give you a preview of what will be covered in the topic and will set your mind in the direction of learning.
- **Exploring the Topic** includes the objectives, concept development, and activities for each objective. Use your own papers to arrive at the answers in the activities.
- **Extra Help** reviews the topic. If you had any difficulty with **Exploring the Topic**, you may find this part helpful.
- **Extensions** gives you the opportunity to take the topic one step further.
- To summarize what you have learned, and to find instructions on doing the unit assignment, turn to the **Unit Summary** at the end of the unit.
- The **Appendices** include the solutions to **Activities (Appendix A)** and any other charts, tables, etc. which may be referred to in the topics **(Appendix B, etc.)**.

Visual Cues

Visual cues are pictures that are used to identify important areas of the material. They are found throughout the booklet.

An explanation of what they mean is written beside each visual cue.

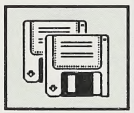


Key Idea
- flagging important ideas

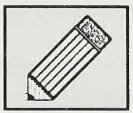
Audiotape
- learning by listening to an audiotape

What You Already Know
- reviewing what you already know

Another View
- exploring different perspectives



Computer Software
- learning by using computer software



Review
- studying previous concepts

Solutions
- correcting the activities

Videotape
- learning by viewing a videotape

Introduction
- introducing the unit

Extra Help
- providing additional study

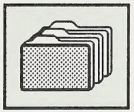
Print Pathway
- choosing a print alternative

What Lies Ahead
- previewing the unit

Extensions
- going on with the topic

Calculator
- using your calculator

Exploring the Topic
- actively learning new concepts



What You Have Learned
- summarizing what you have learned

Mathematics 31

Course Overview

Mathematics 31 contains 9 units. Beside each unit is a percentage that indicates what the unit is worth in relation to the rest of the course. The units and their percentages are listed below. You will be studying the unit that is shaded.

Unit 1 Introduction to Differential Calculus	10%
Unit 2 Differentiation of Algebraic Expressions and Graphing	10%
Unit 3 Practical Application of Derivatives	20%
Unit 4 Integration	10%
Unit 5 Geometric Vectors and Their Application	10%
Unit 6 Algebraic Vectors and Their Application	10%
Unit 7 Inner Product	10%
Unit 8 Systems of Linear Equations	10%
Unit 9 Matrices and Linear Transformations	10%
	100%

Unit Assessment

After completing the unit, you will be given a mark based totally on a unit assignment. This assignment will be found in the Assignment Booklet.

Unit Assignment - 100%

If you are working on a CML terminal, your teacher will determine what this assessment will be. It may be

Unit Assignment - 50%
Supervised Unit Test - 50%

Introduction to Systems of Linear Equations

This unit covers topics dealing with systems of linear equations. Each topic contains explanations, examples, and activities to assist you in understanding systems of linear equations. If you find you are having difficulty with the explanations and the way the material is presented, there is a section called **Extra Help**. If you would like to extend your knowledge of the topic, there is a section called **Extensions**.

You can evaluate your understanding of each topic by working through the activities. Answers are found in the solutions in the **Appendix**. In several cases there is more than one way to do the question.

Unit 8 Systems of Linear Equations

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	Review	8
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	• Introduction • Extra Help • What Lies Ahead • Extensions • Exploring Topic 1	
25%	**Topic 2:** Systems of Linear Equations with Three Variables	34
	• Introduction • Extra Help • What Lies Ahead • Extensions • Exploring Topic 2	
45%	**Topic 3:** Matrices – Definitions and Properties	48
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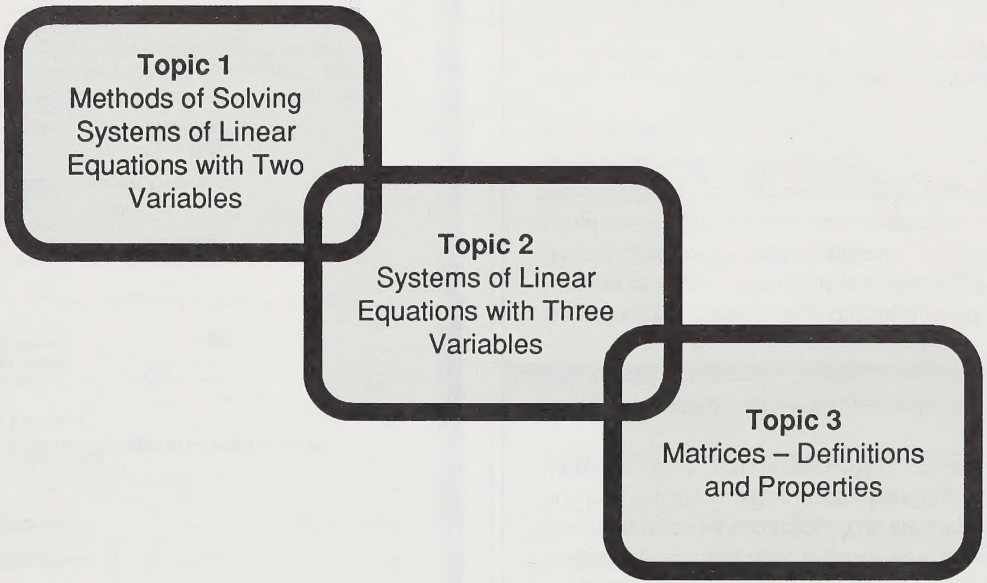
Systems of Linear Equations

In mathematics you need to know how to solve open sentences that contain many variables. You are taking a step forward if you become more skillful in the handling of systems of equations with two, three, or more variables.

The use of graphs is always appropriate for visual interpretations and for verifications.

As you know, mathematical descriptions of practical situations sometimes require two or more variables. For example, if you want to describe the boiling process, you have to consider such variables as time, volume, and temperature. By taking two or three readings, you can answer many questions. You will learn some of these skills in this unit.

Unit 8
Systems of Linear Equations


Topic 1
Methods of Solving Systems of Linear Equations with Two Variables
Topic 2
Systems of Linear Equations with Three Variables
Topic 3
Matrices – Definitions and Properties

What You Already Know

- Recall how to solve an equation with one unknown.

Solve for x in the following equation.

$$5x-9-x=15 \qquad \text{(Add 9 to both sides.)}$$
$$5x-x=15+9 \qquad \text{(Simplify.)}$$
$$4x=24$$
$$x=6$$

You may verify your answer by using the original equation.

Substitute $x=6$ in the original equation.

If the left-hand side is equal to the right-hand side, then your solution is verified.

LS	RS
$5(6)-9-6$	15
$30-15$	15
15	15
LS =	RS

Therefore, $x=6$ is the correct solution.

- Recall how to solve equations with two unknowns (simultaneous equations).

Find the values of x and y that will satisfy these equations.

$$2x+y=21 \quad (1)$$
$$5x-3y=3 \quad (2)$$

First you eliminate one variable from each equation by use of algebraic principles.

$$6x+3y=63 \quad (3)$$
$$5x-3y=3 \quad (2)$$
$$11x=66$$
$$x=6$$

Now that you know $x=6$, you may go back to either equation (1) or (2) to solve for y.

$$2(6)+y=21 \qquad \text{(substituting } x=6 \text{ in (1))}$$
$$12+y=21$$
$$y=9$$

Again, you may verify your answer for $x=6$ and $x=9$.

- Recall how to solve an equation of the second order (quadratic).

Solve the equation $2x^2 - 2x = 112$; then, verify your answer.

$2x^2 - 2x = 112$ (Remove common factors.)

$x^2 - x = 56$ (Set equation equal to 0.)

$x^2 - x - 56 = 0$

At this point you may use either of three methods to find the solution.

– method of factorization

– method of completing the square

– method of the quadratic formula

Use the method of factorization here. Later you might try the other methods to satisfy your own curiosity.

$x^2 - x - 56 = 0$ (Break the middle term into two parts.)

$x^2 - 8x + 7x - 56 = 0$

$(x^2 - 8x) + (7x - 56) = 0$ (Take out the common factor from each set of parentheses.)

$x(x-8) + 7(x-8) = 0$

$(x-8)(x+7) = 0$ (Factor.)

$(x-8) = 0$ or $(x+7) = 0$

$x = 8$ $\qquad$ $x = -7$

Verification:

Substitute $x = 8$ in the original equation.

$2(8)^2 - 2(8) = 112$

$2(64) - 16 = 112$

$128 - 16 = 112$

$112 = 112$

Therefore, $x = 12$ is a solution.

Substitute $x = -7$ into the original equation.

$2(-7)^2 - 2(-7) = 112$

$2(49) + 14 = 112$

$98 + 14 = 112$

$112 = 112$

Therefore, $x = -7$ is a solution.

For $m = (x-8)$ and $n = (x+7)$, if $m \times n = 0$, then either $m = 0$, $n = 0$, or both equal zero.

- Recall how to solve second-order equations with two variables. (simultaneous quadratic).

Solve the following equations for x and y.

$$x^2 + y^2 = 34 \quad (1)$$
$$(x-y)^2 = 4 \quad (2)$$

You must express one variable in terms of the other.

$$\begin{array}{r} x^2 + y^2 - 2xy = 4 \\ x^2 + y^2 = 34 \\ \hline -2xy = -30 \\ xy = 15 \\ y = \frac{15}{x} \end{array} \quad \text{(Subtract (1) from (2).)}$$

Now substitute for y in equation (1).

$$x^2 + \left(\frac{15}{x}\right)^2 = 34$$
$$x^4 + 225 = 34x^2$$
$$x^4 - 34x^2 + 225 = 0$$
$$(x^2 - 9)(x^2 - 25) = 0$$
$$(x-3)(x+3)(x-5)(x+5) = 0$$

Thus, $x = 3, -3, 5,$ or -5.

By substituting $x = 3, -3, 5,$ or -5 in (1) but still satisfying (2), you will find that the solutions are $\{(3,5), (-3,-5), (5,3), (-5,-3)\}$.

Now that you have reviewed this material, go to the **Review** to confirm that you understand the material.

Review

Try the following review questions.

1. Solve for x in the following equations.

   a. $5x+3(8-x)=36$

   b. $7x-4=2x+21$

   c. $2x-5-(x-3)=7$

2. Find the value of x in the following; then verify your answer.

   a. $5(x-2)=25$

   b. $x^2-x+4-(x-1)^2=0$

   c. $6x-4(5x+9)=4(2x-9)$

3. Find the solution to the following equations.

   a. $\frac{1}{3}x+\frac{1}{2}x=10$

   b. $\frac{3x}{4}=\frac{x}{3}+\frac{5}{6}$

   c. $\frac{-(x+5)}{2}+\frac{3x-3}{8}+4=0$

4. Factor the following to find the roots.

   a. $x^2+3x+2=0$

   b. $3x^2+2x=1$

   c. $x^3+21x^2+68x=0$

5. Find the values of x and y; then verify your answer.

   a. $\left.\begin{aligned} x+y&=3\\ x-y&=1\end{aligned}\right\}$

   b. $y-3x=2y-x=20$

   c. $\left.\begin{aligned} x+y&=11\\ \frac{x}{2}+\frac{y}{3}&=5\end{aligned}\right\}$

6. Find the value of the unknowns in each of the following.

a. $\left.\begin{aligned} x^2+y^2&=130\\ x^2-y^2&=32 \end{aligned}\right\}$

b. $\left.\begin{aligned} (x+y)^2&=144\\ xy&=32 \end{aligned}\right\}$

c. $\left.\begin{aligned} a^2-b^2&=11\\ a+b&=11 \end{aligned}\right\}$

7. Solve this system of linear equations.

$$\left.\begin{aligned} x+y&=10\\ y+z&=8\\ x+z&=12 \end{aligned}\right\}$$

Now go to the **Review** solutions in **Appendix A.**

Topic 1 Methods of Solving Systems of Linear Equations with Two Variables

Introduction

When you solve linear equations with two unknowns, you know that you need at least two equations with the same variables in order to find the solution. The two equations are referred to as a system, since the variables of the one will also satisfy the other equation.

You can therefore solve the system of equations for the two unknowns. As a rule of thumb you can say that the number of equations needed to solve a system of equations is equal to the number of different variables in the system. In other words if you have ten variables in the system, you need to have ten equations in order to solve the system for the variables.

In this topic you will learn how to solve systems of equations using a step by step approach in addition to other methods.

What Lies Ahead

In this topic you will learn to

1. solve systems of linear equations with two unknowns
2. classify systems of linear equations in such categories as independent, dependent, or inconsistent

Now that you know what to expect, turn the page to begin your study of methods of solving systems of linear equations with two variables.

Exploring Topic 1

Activity 1

Solve systems of linear equations with two unknowns.

A first-degree equation in one variable will result in a straight line on a graph. The line will either be horizontal or vertical. For example, $ax+b=0$ will give a vertical line and $ay+b=0$ will give a horizontal line.

A first-degree equation in two variables will result in a straight oblique line when graphed on a grid. This line may have a positive or a negative slope. For example, $y=mx+c$ where $m>0$ will give a line that is slanted to the right, and $y=mx+c$ where $m<0$ will give a line that is slanted to the left.

In this study you will concern yourself more with the value or values that satisfy the equations rather than the slopes of these lines.

If you have two first-degree equations in two variables, you can solve for the variables x and y that will satisfy both equations simultaneously.

If the solutions can be found, then you have a pair of equations called a system of two equations. The values of x and y which satisfy the equations are called the solutions of the system of equations.

The solution represents the ordered pair of the point at which the lines intersect.

You could very well draw the lines on a grid and arrive at the point of intersection; however, you will arrive at this point analytically by solving the two equations.

It is important for you to remember that you need at least two equations to solve a first-degree equation in two variables.

Three methods of solving a first-degree equation in two variables are the following:

- **comparison method**
- **substitution method**
- **elimination method**

You will now try a few examples using the comparison method. When using the comparison method, you first seek to obtain expressions for one of the variables from both equations and then compare them in order to obtain one equation in one variable. Secondly, solve the equation for the value of one variable, and then go back to one of the two original equations and solve for the other variable.

Study the following example.

Example 1

Use the comparison method to solve and verify the system of equations.

$x+2y=6$ (1)
$x+y=2$ (2)

Solution:

Manipulate (1) and (2) to read as follows:

$$x=6-2y \quad (1)$$
$$x=2-y \quad (2)$$
$$\therefore 6-2y=2-y \quad \text{(since they equal } x\text{)}$$
$$2y-y=6-2$$
$$y=4$$

Since $y=4$, you can find x by substituting in (1).

$$x+y=2$$
$$x+4=2$$
$$x=2-4$$
$$=-2$$

Verification for (1):

LS	RS
$x+2y$	6
$-2+2(4)$	6
$-2+8$	6
6	6
LS =	RS

Verification for (2):

LS	RS
$x+y$	2
$-2+4$	2
2	2
LS =	RS

The verification is complete when the left side is the same as the right side.

Now examine another similar example.

Example 2

Use the comparison method to obtain the solution to the following system of equations and verify your answer.

$2x+y=3$ (1)
$5x+2y=7$ (2)

Solution:

Manipulate (1) and (2) to read as follows:

$$y=3-2x$$
$$y=\frac{7-5x}{2}$$

$$\therefore 3-2x=\frac{7-5x}{2} \quad \text{(since both sides equal } y\text{)}$$

$$6-4x=7-5x$$

$$5x-4x=7-6$$

$$x=1$$

Substitute $x=1$ in (1) and solve for y.

$$2x+y=3$$

$$y=3-2(1)$$

$$=3-2$$

$$=1$$

Verification for (1):

LS	RS
$2x+y$	3
$2(1)+(1)$	3
3	3
LS =	RS

Verification for (2):

LS	RS
$5x+2y$	7
$5(1)+2(1)$	7
7	7
LS =	RS

Another way of solving a system of linear equations is by the method of substitution. This method allows you to express one variable in terms of another and then solve the new equation which has only one variable. Study the following example.

Example 3

Solve the following system of equations using the substitution method.

$$x+y=3 \quad (1)$$

$$x+2y=4 \quad (2)$$

Solution:

Manipulate (2) to read as follows:

$$x+y+y=4 \quad (3)$$

Now substitute $x+y=3$ in (3).

$$3+y=4$$

$$y=1$$

Substitute $y=1$ in (1).

$$x+1=3$$

$$x=3-1$$

$$=2$$

Therefore, $x=2$ and $y=1$.

Another way of looking at the method of substitution is in terms of ordered pairs.

Consider the equations $x+y=3$ and $x+2y=4$ again. If you express the solution in terms of ordered pairs (x_1, y_1), then y_1 can be expressed in terms of x_1.

You can write $x_1+y_1=3$ or $y_1=3-x_1$, and the ordered pair (x_1, y_1) becomes $(x_1, 3-x_1)$.

You can find the solution to the system by substituting this new ordered pair in the equation and solving for the one unknown. When the value of the one unknown is determined, go back and substitute in order to determine other unknowns.

Substitute $y=3-x$ in equation (2), and then solve for x.

$$\begin{aligned} x+2(3-x)&=4\\ x+6-2x&=4\\ x-2x&=4-6\\ -x&=-2\\ x&=2 \end{aligned}$$

Substitute $x=2$ in either (1) or (2) to calculate the value of y.

It should be mentioned here that the steps of the argument are reversible; you could write x in terms of y.

Now try another example.

Example 4

Use the method of substitution to find the value of x and y in this system of equations.

$2x=y-2$ (1)
$x-2y=5$ (2)

Solution:

Manipulate (1) to read $y=2x+2$.

Hence the ordered pair (x_1, y_1) becomes $(x_1, 2x+2)$.

Substitute for y in (2).

$$\begin{aligned} x-2(2x+2)&=5\\ x-4x-4&=5\\ -3x&=5+4\\ 3x&=-9\\ x&=-3 \end{aligned}$$

Substitute $x=-3$ in (1).

$$\begin{aligned} 2x&=y-2\\ 2(-3)&=y-2\\ y&=2-6\\ y&=-4 \end{aligned}$$

Therefore, $x=-3$ and $y=-4$.

The third way of solving systems of equations is by the elimination method. Here you add or subtract the two equations in order to eliminate one of the two variables. A new equation in one variable is obtained, and it is usually easy to solve an equation with one variable.

Example 5

Solve the following system of equations by using the elimination method.

$2x+y=5$ ①
$3x-y=20$ ②

Solution:

$$\begin{array}{rr} & 2x+y=5 \\ & 3x-y=20 \\ \hline ①+②: & 5x=25 \\ & x=5 \end{array}$$

Substitute $x=5$ in ①.

$$\begin{aligned} 2x+y&=5 \\ 2(5)+y&=5 \\ y&=5-10 \\ &=-5 \end{aligned}$$

You may wish to verify your results by substituting $x=5$ and $y=-5$ into the original equations.

Verification for ①:

LS	RS
$2x+y$	5
$2(5)+(-5)$	5
5	5
LS =	RS

Verification for ②:

LS	RS
$3x-y$	20
$3(5)-(-5)$	20
20	20
LS =	RS

Notice that the elimination method works very well when one variable can be eliminated directly. However, you can also use it when neither of the variables has the same numerical coefficient. In this case you have to set up an equivalent system in which the coefficients of either x or y are equal.

Example 6

Use the method of elimination to solve the following system of equations.

$3x-2y-5=0$ (1)
$4x-3y-11=0$ (2)

Solution:

Make the numerical coefficients of either variable equal in magnitude but opposite in sign.

$4\times$(1): $12x-8y-20=0$ (3)
$-3\times$(2): $-12x+9y+33=0$ (4)
(3) + (4): $y+13=0$
$y=-13$

Substitute $y=-13$ in (3).

$3x-2(-13)-5=0$
$3x+26-5=0$
$3x=-21$
$x=-7$

Therefore, $x=-7$ and $y=-13$.

Verification for (1):

LS	RS
$3x-2y-5$	0
$3(-7)-2(-13)-5$	0
$-21+26-5$	0
0	0
LS =	RS

Verification for (2):

LS	RS
$4x-3y-11$	0
$4(-7)-3(-13)-11$	0
$-28+39-11$	0
0	0
LS =	RS

Therefore, the solution $x=-7$ and $y=-13$ is correct.

Now try one example that is a bit more difficult.

Example 7

Solve the following system of equations, and verify your answer.

$\frac{x}{3}+\frac{y}{4}=6$ (1)

$y-\frac{x-y}{4}=7$ (2)

Solution:

Multiply (1) by 12 and (2) by 4.

$12\times$(1): $4x+3y=72$ (3)

$4\times$(2): $-x+5y=28$ (4)

Now multiply (4) by 4 and add to (3).

$4\times$(4): $-4x+20y=112$ (5)

(5) + (3): $4x+3y=72$ (3)

$23y=184$

$y=8$

Now substitute $y=8$ in (4).

$-x+5(8)=28$

$-x+40=28$

$-x=28-40$

$x=12$

Therefore, $x=12$ and $y=8$.

Verification for (1):

LS	RS
$\frac{12}{3}+\frac{8}{4}$	6
$4+2$	6
6	6

LS = RS

Verification for (2):

LS	RS
$8-\frac{12-8}{4}$	7
$8-\frac{4}{4}$	7
$8-1$	7
7	7

LS = RS

So far you have covered the comparison method, the substitution method, and the elimination method. When you have a system of equations to solve, you may need to use one or more of these methods. Very often it is quite apparent which method to use. With some experience you will be able to tell which one to use.

Do the following practice exercise.

Try the following problems.

1. Use the comparison method to find the value of the variables in the systems of equations.

   a. $\left.\begin{aligned} 2x-y&=7 \\ 5x+2y&=13 \end{aligned}\right\}$

   b. $\left.\begin{aligned} y&=2x+3 \\ x+y&=6 \end{aligned}\right\}$

2. Use the substitution method to solve the systems of equations.

   a. $\left.\begin{aligned} x+2y&=15 \\ 3x-2y&=9 \end{aligned}\right\}$

   b. $\left.\begin{aligned} 2x-5y&=6 \\ 5x+2y&=\frac{1}{2} \end{aligned}\right\}$

3. Use the elimination method to solve the systems of equations.

   a. $\left.\begin{aligned} x-2y&=-3 \\ 2x+y&=-2 \end{aligned}\right\}$

   b. $\left.\begin{aligned} x+2y&=15 \\ 6x-4y&=18 \end{aligned}\right\}$

4. Solve the following systems of equations, and verify your answer.

   a. $\left.\begin{aligned} 3x&=2y \\ 12y&=-8x+13 \end{aligned}\right\}$

   b. $\left.\begin{aligned} x+\frac{1}{3}y&=y \\ y+\frac{1}{4}x&=x+3 \end{aligned}\right\}$

For solutions to **Activity 1**, turn to **Appendix A, Topic 1**.

Activity 2

Classify systems of linear equations in such categories as independent, dependent, or inconsistent.

When you have a system of linear equations with two (or more) variables, they can be written in another equivalent form called an **equivalent reduced system.**

The two systems are equivalent because they have the same solution, and they are reduced because the second equation has only one variable.

Example 8

Obtain an equivalent reduced system for the following system of equations.

$5x+2y=3$ (1)
$2x-3y=5$ (2)

Solution:

Multiply (1) by 2 and (2) by 5.

$$\begin{aligned} 2\times(1): \quad & 10x+4y=6 \quad (3)\\ 5\times(2): \quad & 10x-15y=25 \quad (4)\\ (3)-(4): \quad & 19y=-19\\ & y=-1 \end{aligned}$$

Substitute $y=-1$ in (1).

$$\begin{aligned} 5x+2y&=3\\ 5x+2(-1)&=3\\ 5x&=3+2\\ x&=1 \end{aligned}$$

Therefore, $x=1$ and $y=-1$.

When you reduced the original system to $\left.\begin{matrix}5x+2y=3\\ y=-1\end{matrix}\right\}$, you have reduced the given system to an equivalent reduced system of the equations.

Once an equivalent reduced system of equations is obtained, the solution is clear. Using the substitution method you can easily determine the other variable.

It is not always possible to determine the solution to a system of equations because the system may have a unique solution, an infinite number of solutions, or no solution at all.

Since you can classify systems of equations according to their solution, you will therefore have three types of systems.

Independent Systems of Equations

Consider the following system of equations.

$x+2y=7$ (1)
$2x+y=3$ (2)

The system has only one value for x and one for y that will satisfy the system of equations. The calculations for obtaining x and y are shown.

$$-2\times(1): \quad -2x-4y=-14 \quad (3)$$
$$\underline{\qquad 2x+y=3 \qquad (2)}$$
$$(3)+(2): \quad -3y=-11$$
$$y=\frac{-11}{-3}$$
$$=\frac{11}{3}$$

Substitute $y=\frac{11}{3}$ in (1).

$$x+2\left(\frac{11}{3}\right)=7$$
$$x=\frac{-22}{3}+7$$
$$=-\frac{1}{3}$$

The solution is $x=-\frac{1}{3}$ and $y=\frac{11}{3}$.

These values are the only ones that will satisfy the system. This system has a **unique solution**.

When a system of equations has one solution for x and y, the system is said to be independent.

As a rule of thumb you might say that if the respective coefficients of at least x and y in the second equation are not multiples of those in the first equation, then the equations are independent.

Dependent Systems of Equations

If a system of equations has an infinite number of solutions, then the equations are said to be dependent.

Consider the following system of equations.

$x-3y=2$ (1)
$3x-9y=6$ (2)

If you try to solve this system, you will discover that there are an infinite number of solutions for the equations of the system.

Try solving (1) and (2) for x and y.

$3 \times (1)$: $3x - 9y = 6$ (3)

$3x - 9y = 6$ (2)

You will see that two equations are the same. If you add or subtract the equations, you do not arrive at a unique solution. There is either no equation at all or an equation with infinite solutions.

Look at the case when you subtract the equations.

$3x - 9y = 6$ (3)

$3x - 9y = 6$ (2)

(3) $-$ (2): $0x + 0y = 0$

Look at the case when you add the equations.

$3x - 9y = 6$ (3)

$3x - 9y = 6$ (2)

(3) $+$ (2): $6x - 18y = 12$ (4)

Multiply (4) by $\frac{1}{6}$.

$\frac{1}{6} \times (4)$: $x - 3y = 2$

The result is an equation with two variables. This equation is dependent since the variable x depends on the variable y, and vice versa, in order to make the equation true.

Example 9

Assign three values to x and y to make $x - 3y = 2$ true.

Solution:

Let $x = 5$ and $y = 1$.

$$x - 3y = 2$$
$$(5) - 3(1) = 2$$
$$5 - 3 = 2$$
$$2 = 2$$

Therefore, $x = 5$ and $y = 1$ is a solution.

Let $x = 8$ and $y = 2$.

$$x - 3y = 2$$
$$(8) - 3(2) = 2$$
$$8 - 6 = 2$$
$$2 = 2$$

Therefore, $x = 8$ and $y = 2$ is a solution.

Let $x = -4$ and $y = -2$.

$$x - 3y = 2$$
$$-4 - 3(-2) = 2$$
$$-4 + 6 = 2$$
$$2 = 2$$

Therefore, $x = -4$ and $y = -2$ is a solution.

As you can see these hand picked values will satisfy the system of equations. There are countless other such ordered pairs. This is why equations of this form are sometimes called indeterminate.

Historically, this kind of equation has been of particular interest when the coefficients are integers and it is required to find expressions for the set of integral values of the variables that satisfy the given equation. Systems under these restrictions are called **Diophantine** equations.

A dependent system of equations occurs when all the terms of one equation are multiples of the respective terms of the other equation.

Inconsistent Systems of Equations

When a system of equations has no solution, the solution set is said to be empty. The empty set is generally designated by either the symbol ϕ or $\{\ \}$. Such systems of equations are said to be inconsistent.

Consider the following system of equations.

$x+2y=3$ (1)
$x+2y=6$ (2)

There are no values of x and y that will satisfy the equations simultaneously. If you try any algebraic maneuvers in an attempt to find a solution, you will find that both variables are eliminated each time.

Subtract (1) from (2).

$$\begin{array}{rl} & x+2y=3 \quad (1) \\ & x+2y=6 \quad (2) \\ \hline (1)-(2): & 0x+0y=-3 \end{array}$$

The result is not meaningful. There are no values for x and y that will satisfy the system of equations.

If the coefficients of x and y in the second equation are the same as or a multiple of those in the first equation but the constant terms are not the same, then the system of equations is inconsistent.

If there are more than two variables in the equations of the system, it is not quite so easy to tell what type of system exists. You will learn later in this course how to deal with equations with three or more variables.

You will now take a look at some examples dealing with systems of equations and see how they can be represented graphically.

Example 10

Graph the following system of equations, and state the classification of the system.

$3x+6y=9$ (1)
$4x+3y=2$ (2)

Solution:

Change the system to an equivalent reduced system.

$\frac{4}{3}\times$(1): $4x+8y=12$ (3)

$4x+3y=2$ (2)

(3) − (2): $5y=10$

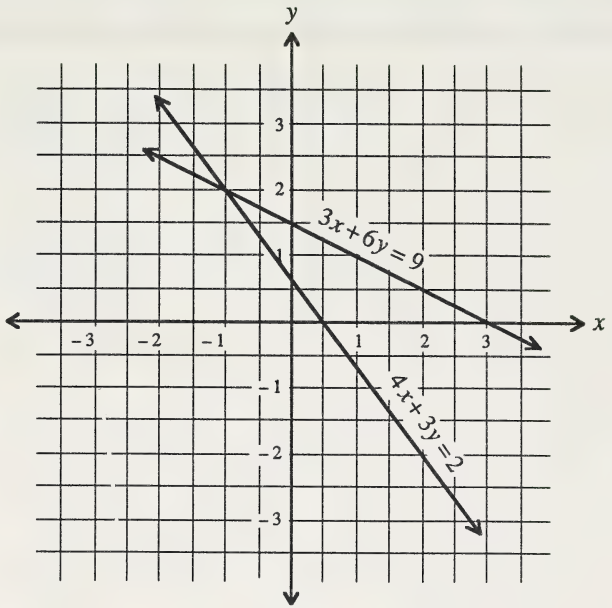
$y=2$

The following is an equivalent reduced system.

$$\left.\begin{aligned} 3x+6y&=9\\ y&=2\end{aligned}\right\}$$

Now substitute $y=2$ in (1) and solve for x.

$3x+6(2)=9$
$3x+12=9$
$3x=-3$
$x=-1$



The solution to the system is $x=-1$ and $y=2$.

The graphical representation shows that the two lines intersect at $(-1,2)$. This verifies that the solution to the system is indeed $x=-1$ and $y=2$.

The solution is unique to this system; therefore, the system is independent.

Example 11

Examine the following system of equations.

$2x-y=13$ (1)
$2x-y=-10$ (2)

Find the solution if it exists, and state the type of system that is represented by these equations.

Solution:

Change the original system to an equivalent reduced system

$$\begin{array}{rll} & 2x-y=13 & (1) \\ & 2x-y=-10 & (2) \\ \hline (1)-(2): & 0x+0y=23 & (3) \end{array}$$

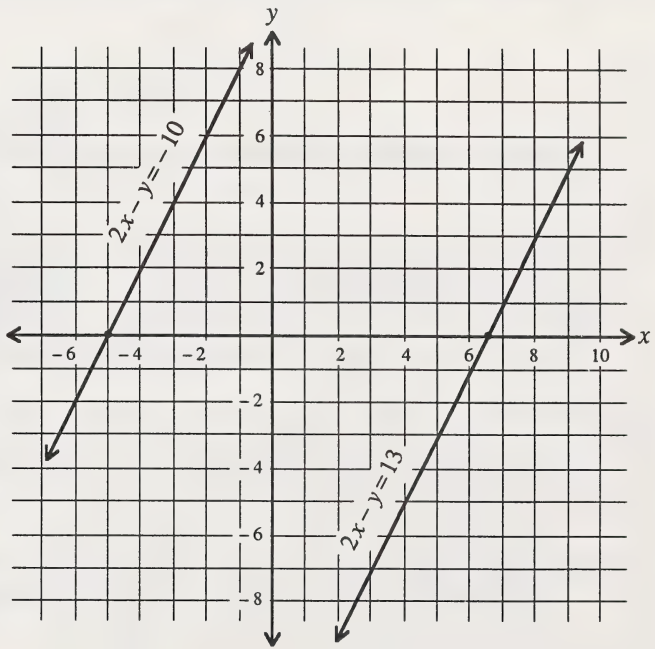
This is an equivalent reduced system.

Since the left-hand side of (3) is zero and the right-hand side is 23, the equation is not a true one. Hence, the solution is ϕ (the empty set), and the system is inconsistent.

The reason for writing $0x$ and $0y$ on the left side of (3), rather than just zero or leaving it blank, is to retain the variables x and y. It becomes more evident that both sides of the equation are not compatible.

Since $0x$ is equal to zero for all real values of x and the same for $0y$, you realize that there are no values for x and y which will make the left-hand side equal to 23.

A graphical representation of Example 11 is as follows:



$2x-y=13$

x	0	2	4	6.5
y	−13	−9	−5	0

$2x-y=-10$

x	−5	−3	−1	0
y	0	4	8	10

The system of equations results in two parallel lines. Note on the graph that the lines have the same slope but different x and y intercepts.

You can now say that if the graph of the equations of a system produces parallel lines, then the system is inconsistent.

Example 12

Study the following system of equations.

$2x+y=3$ (1)
$4x+2y=6$ (2)

Graph each equation, and state whether the system is dependent, independent, or inconsistent.

Solution:

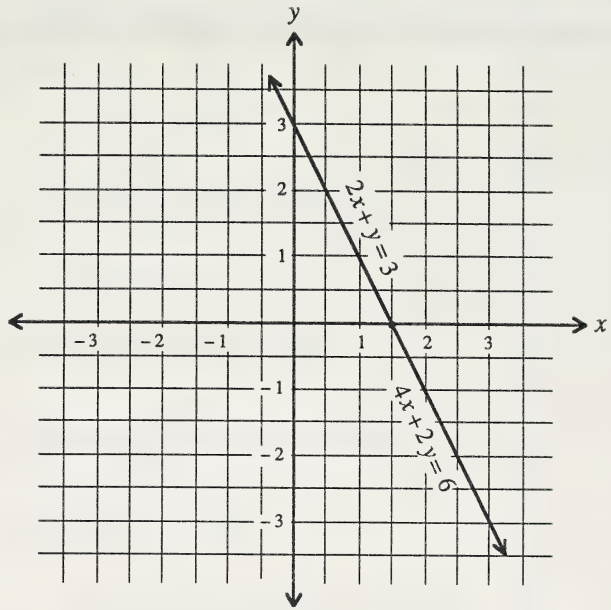
Change the system to an equivalent reduced system.

$2\times(1)$: $4x+2y=6$ (3)

$$\begin{array}{rl} & 4x+2y=6 \quad (2) \\ \hline (3)-(2): & 0x+0y=0 \end{array}$$

The following is an equivalent reduced system.

$$\left.\begin{array}{l} 4x+2y=6 \\ 0x+0y=0 \end{array}\right\}$$



The graph of this system is one line on top of another.

Since the two equations have the same line, there is an infinite set of solutions for the system. When this is the case, the system is dependent.

You can now state that if a common straight line results when a system of equations is graphed, then the system is dependent.

There is another way of looking at dependent systems.

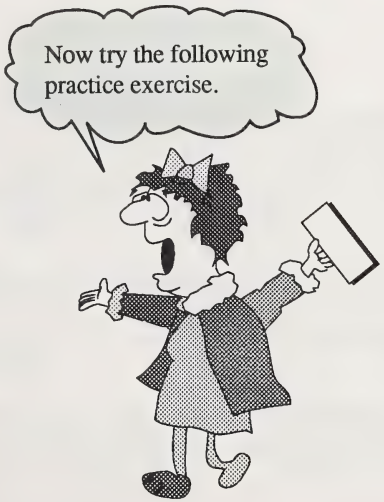
Let y have the value k (where $k \in R$). Then, write (1) in the last example as $2x + k = 3$, and solve for x.

$$2x = 3 - k$$
$$x = \frac{3-k}{2}$$

The system has an infinite number of solutions given by $\left(\frac{3-k}{2}, k\right)$ since $k \in R$.

For specific examples let $k = 0$, 1, or any other real number. Try $k = 0$. If $\left(\frac{3-k}{2}, k\right)$ is a true statement, then $\left(\frac{3}{2}, 0\right)$ is a solution.

Try $k = 1$. If $\left(\frac{3-k}{2}, k\right)$ is true, then (1, 1) is a solution.



Do any four of the following problems.

1. Find the solution to the system of equations by placing the correct numbers in the blank spaces. (Convert the system to an equivalent reduced system.)

$3x + 7y = 24$ (1)
$5x + 2y = 11$ (2)

$2 \times$ (1): ___ x + ___ y = ___ (3)
$7 \times$ (2): ___ x + ___ y = ___ (4)

(3) − (4): ___ x = ___
x = ___

Substitute $x =$ ___ in (1).

3___ + $7y = 24$
$y =$ ___

The solution is $x =$ ___ and $y =$ ___.

2. Study the following system of equations.

$2x-5y=6$ (1)

$5x+2y=\frac{1}{2}$ (2)

a. Solve the system of equations by first changing it to an equivalent reduced system.

b. Sketch the graph.

c. State the type of system.

3. Study the following system of equations.

$x-2y=-3$ (1)

$2x+y=-2$ (2)

a. Solve the system by first changing it to an equivalent reduced system.

b. Sketch the graph.

c. State the type of system.

4. Study the following system of equations.

$x+2y=5$ (1)

$2x+4y=6$ (2)

a. Solve the system by first changing it to an equivalent reduced system.

b. Sketch the graph.

c. State the type of system.

5. Study the following system of equations.

$2x-3y=5$ (1)

$6x-9y=15$ (2)

a. Solve the system by first changing it to an equivalent reduced system.

b. Sketch the graph.

c. State the type of system.

For solutions to **Activity 2**, turn to **Appendix A, Topic 1.**

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section. } You may decide to do both.

Extra Help

The equation of a straight line in standard form is $Ax+By+C=0$ where x and y are variables that can take on the value of any real number. A, B, and C are real numbers where both A and B cannot equal zero. When you have this situation, you can draw a graph. The graph will be a straight line; hence, this represents a linear equation. If you have two linear equations, you can often find the solution to the equations.

When there are two linear equations which relate to each other in certain ways, then this is a system of equations.

If the two equations have a common solution, the equations are said to be independent.

Consider the following system of equations.

$x-y+2=0$ (1)

$2x+y+4=0$ (2)

Solve the equations by adding the two equations.

$$\begin{array}{rr} x-y+2=0 & (1) \\ 2x+y+4=0 & (2) \\ \hline 3x+6=0 & \\ x=-2 & \end{array}$$

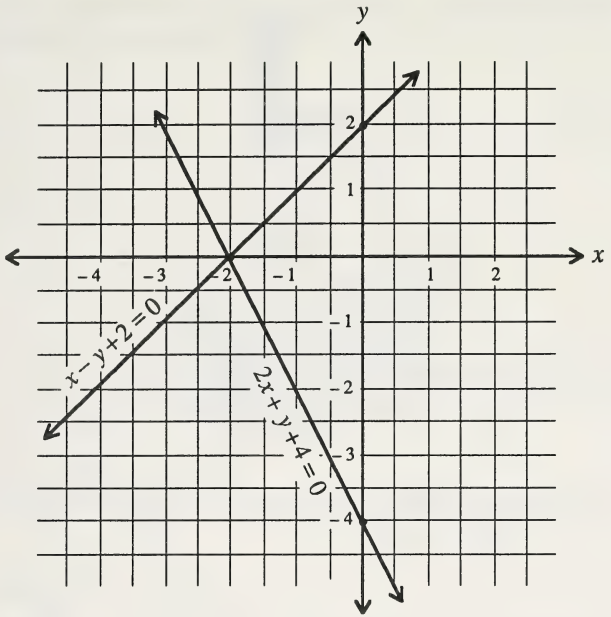
(1) + (2):

Substitute $x=-2$ in (1).

$$-2-y+2=0$$
$$y=0$$

The two lines intersect at $(-2, 0)$.

Examine the graph of these two lines.



The intersection point agrees with the calculation for the two equations.

You can draw the conclusion that if a system of two linear equations has exactly one solution for x and y, then the system is independent.

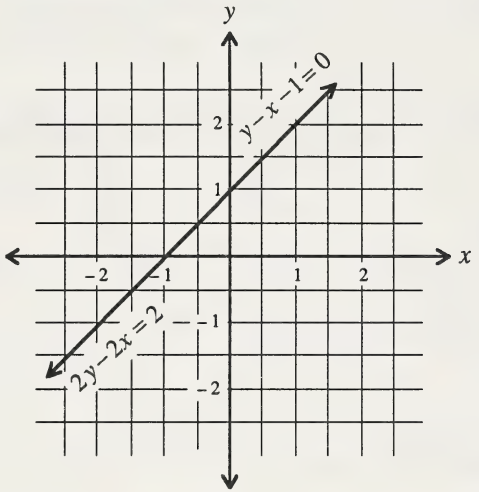
Consider the following system of equations.

$y - x - 1 = 0$ (1)

$2y - 2x = 2$ (2)

If equation (1) is multiplied by 2, then it becomes equation (2). You cannot eliminate either variable. One equation is simply multiplied through by a factor to get the other equation. They are equivalent, and they have the same graph. This means that an infinite number of solutions exist for these two equations. For example, $(-1, 0)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$ are solutions to this system along with many others.

Examine the graph of these two lines.



The graph reveals the two lines are running concurrently. When this is true of two equations, you can say that the system of equations is dependent.

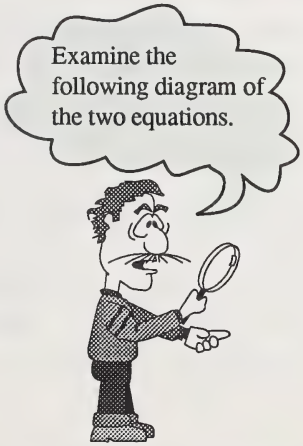
There is still one system that must be examined. This is the system where there is no solution. Consider the following system of equations.

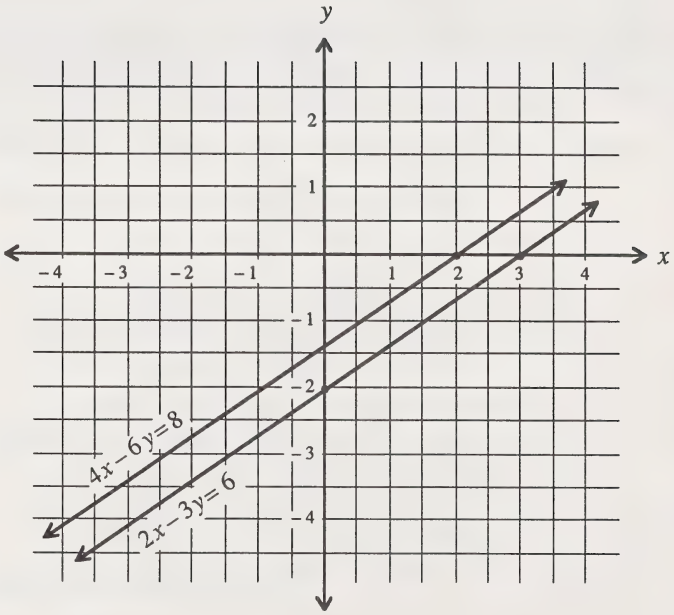
$2x-3y=6$ (1)
$4x-6y=8$ (2)

You can try to solve for x or y by multiplying (1) by 2 and then subtracting (2) from the new equation.

$2\times(1)$: $4x-6y=12$ (3)
$4x-6y=8$ (2)
(3) − (2): $0x+0y=4$

This result shows that there is no value for x and y that will satisfy both (1) and (2).





A system of equations which has the empty set for its solution is called an inconsistent system.

In summary you see that there are three possible types of systems of equations. They may either be dependent, independent, or inconsistent. The graph as well as the solution will determine the type of system.

Now try the following exercise which deals with systems of equations.

Do the following problems.

1. Sketch the following systems of equations, and classify each as dependent, independent, or inconsistent.

   a. $\left.\begin{aligned} 2y &= x+1 \\ 2x+2 &= 4y \end{aligned}\right\}$

   b. $\left.\begin{aligned} x &= y \\ x &= 2y \end{aligned}\right\}$

   c. $\left.\begin{aligned} x+y &= 0 \\ 2x+3y &= 3 \end{aligned}\right\}$

   d. $\left.\begin{aligned} x-2y &= 0 \\ 2y-x &= 2 \end{aligned}\right\}$

2. For the following systems, solve for x and y; then, graph the equations, and classify each system as independent, inconsistent, or dependent.

   a. $\left.\begin{aligned} x+2y &= 3 \\ 3y+4x &= 2 \end{aligned}\right\}$

   b. $\left.\begin{aligned} 3x+y &= 1 \\ 2y+6x &= 2 \end{aligned}\right\}$

For solutions to **Extra Help**, turn to **Appendix A, Topic 1**.

Extensions

Steps for Problem Solving

Systems of equations are especially useful when you need to solve more tedious problems. Try to follow the following steps for problem solving.

Step 1: Read the problem through very carefully. Then, think about the following questions.

- What information is given here?
- What information is asked for?

Step 2: Translate English sentences into mathematical language such as algebraic expressions and equations.

Step 3: Solve equations where possible.

Step 4: Verify your answer by checking it with the original problem.

Step 5: Write a final statement expressing your answer as clearly and as precisely as you can. Be sure to include the units where applicable.

If you end up with two variables, you need to solve a system of equations. That is, you must have at least two equations.

Now examine some problem solving exercises that lead to system of equations.

Example 13

Two numbers differ by 25. If three times the smaller exceeds the larger by 3, find the two numbers and verify your answer.

Solution:

Let the larger number be x and the smaller number be y.

$$\begin{aligned} x - y &= 25 \quad (1) \\ 3y - x &= 3 \quad (2) \\ (1)+(2): \quad 3y - y &= 28 \\ 2y &= 28 \\ y &= 14 \end{aligned}$$

Substitute $y = 14$ in (1).

$$\begin{aligned} x - 14 &= 25 \\ x &= 25 + 14 \\ &= 39 \end{aligned}$$

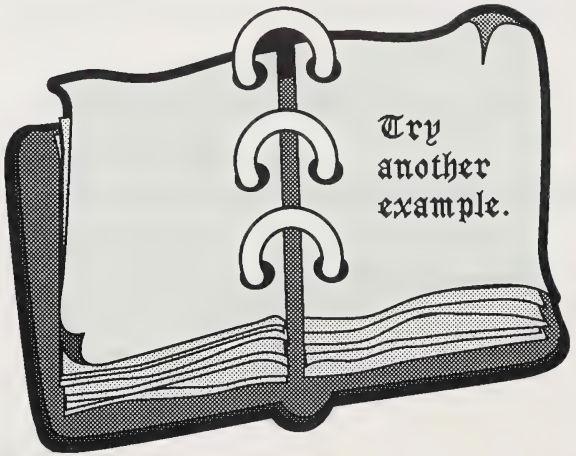
The larger number is 39 and the smaller number is 14.

Verification for (1):

LS	RS
$x - y$	25
$39 - 14$	25
25	25
LS =	RS

Verification for (2):

LS	RS
$3y - x$	3
$3(14) - 39$	3
3	3
LS =	RS



Example 14

The Kiosk Store cash register had $30 in loonies and two-dollar bills. There are three times as many loonies as there are two-dollar bills. How many of each type are there?

Solution:

Let x represent the number of loonies, and let y represent the number of two-dollar bills.

$$\begin{aligned} 3y - x &= 0 \quad (1) \\ x + 2y &= 30 \quad (2) \\ (1)+(2): \quad 3y + 2y &= 30 \quad (3) \\ 5y &= 30 \\ y &= 6 \end{aligned}$$

Substitute $y = 6$ in (2).

$$\begin{aligned} x + 2y &= 30 \\ x + 2(6) &= 30 \\ x &= 30 - 12 \\ &= 18 \end{aligned}$$

There are 18 loonies and 6 two-dollar bills.

You can use systems of equations to show that statements are false, inconsistent, or true.

The following are problem-solving exercises. Follow the five steps outlined earlier to solve them.

Do any three of these problems.

1. Heidi and Dimetri bought lottery tickets and won $1000. Since they paid different amounts for the winning ticket, Heidi received $100 more than twice what Dimetri got. How much did each person receive?

2. A boy told his teacher that the difference between his age and his father's was 20 years. He also said that twice his father's age less 40 years was equal to twice his own age. Show that the boy's statements were consistent but redundant.

3. The perimeter of a rectangular plot of land measures 98 m, and its diagonal is 41 m. Find the length and width of the plot.

4. The profits for one month from a garage totalled $1200. The treasurer deposited part in an account earning 13% interest and the rest in another account earning 4% interest. If the total interest earned over a year was $120, how much money was deposited in each account?

5. A cleaning agent containing 20% ammonia was mixed with another agent containing 10% ammonia to dilute its strength. If 100 L of 17% ammonia was obtained from the mixture, how much of each agent was used?

For solutions to **Extensions**, turn to **Appendix A, Topic 1**.

Topic 2 Systems of Linear Equations with Three Variables

Introduction

A vertical line or a horizontal line can be represented by one variable only. A plane can be defined in terms of two variables. It should not be surprising to you to know that if you wish to define three-space, you will need three variables in three equations.

If you want to know the value of these variables, you need to solve a system of three equations. You will find out how this is done in the section.

What Lies Ahead

Throughout the topic you will learn to

1. define systems of linear equations as homogeneous or nonhomogeneous and solve the systems

Now that you know what to expect, turn the page to begin your study of systems of linear equations with three variables.

Exploring Topic 2

Activity 1

Define systems of linear equations as homogeneous or nonhomogeneous and solve the systems.

In order to solve a system of equations with three variables, you have to use the methods you learned in the last section. That is, you use the methods of comparison, substitution, and elimination. Since there are three variables, you may have to use some of these methods more than once before you can eliminate certain variables.

As a rule, one linear equation with two variables will have an infinite number of solutions. Two linear equations with two variables usually define a single ordered pair. In other words, they have one common point (if the system is independent).

In order to solve systems with three equations, you seek to eliminate variables to obtain equations with fewer variables until you find a value for each variable.

Look at the following example.

Example 1

Find three numbers such that when you subtract the first from the second you get 1, when you add the first and the third you get 4, and when you add twice the second to the third you get 7.

Solution:

Let x be the first number, let y be the second number, and let z be the third number.

$$y-x=1 \quad (1)$$
$$x+z=4 \quad (2)$$
$$2y+z=7 \quad (3)$$

$$\begin{array}{lrl} & 2y+z=7 & (3) \\ (1)+(2): & y+z=5 & (4) \\ (3)-(4): & y=2 & \end{array}$$

Substitute $y=2$ in (1).

$$y-x=1$$
$$2-x=1$$
$$x=1$$

Substitute $x=1$ in (2).

$$x+z=4$$
$$1+z=4$$
$$z=3$$

Therefore, the three numbers are 1, 2, and 3.

The method outlined in the previous example is by no means unique. You may follow any pattern to eliminate variables. You simply aim to reduce the amount of work you have to do.

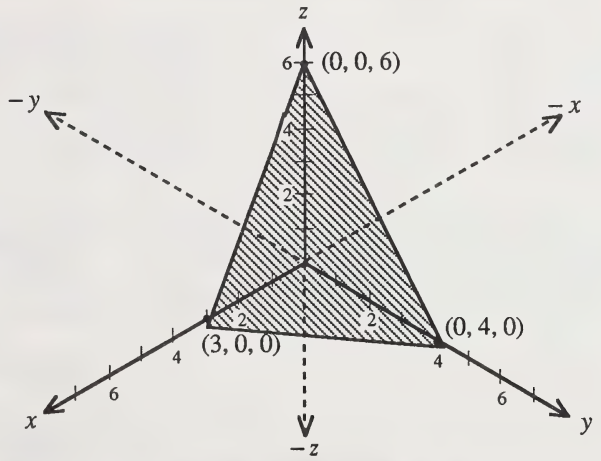
The graphical interpretation of an equation in three variables is a set of points in three-dimensional space. Three coordinates are required to determine a point in space with reference to three axes.

A single equation with three variables defines a plane in space.

Consider the following equation.

$4x+3y+2z=12$

This equation is defining the plane that passes through the points in space $(3,0,0)$, $(0,4,0)$, and $(0,0,6)$. These points are found by letting two variables be equal to zero and then calculating the third. The plane $4x+3y+2z=12$ should look like this figure:



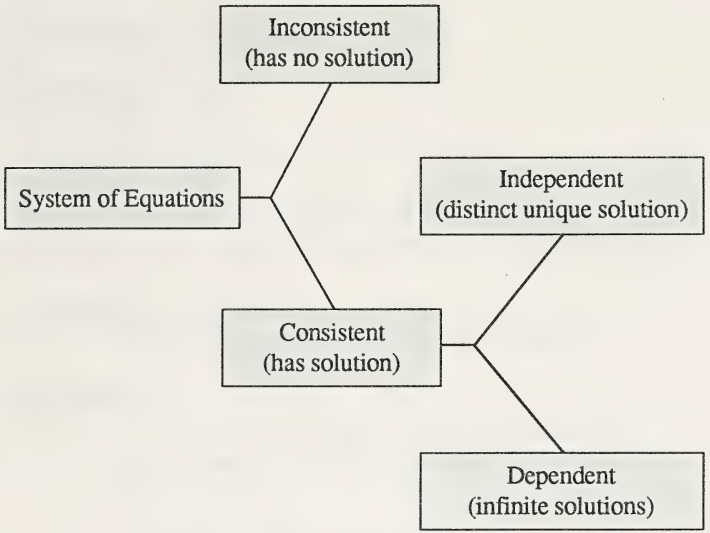
A system of three linear equations in three variables may represent planes having one of the following representations in three-dimensional space.

- a single plane
- two parallel planes
- three parallel planes
- two distinct planes intersecting in one line
- three distinct planes intersecting in pairs in a set of three parallel lines
- two parallel planes and a third plane intersecting the first two in a pair of parallel lines
- three distinct planes intersecting in a straight line
- three distinct planes intersecting in a single point

These are all the possible geometric orientations of three planes in three-space.

In Topic 1 you learned about the three possible systems of equations, namely dependent, independent, and inconsistent systems. A system of equations with three variables can be classified in the same manner.

You shall use the following diagram of linear equations to determine the system according to the solutions.



If a system of equations with three variables has a unique solution, the system is independent.

Try an example which demonstrates this.

Example 2

Solve the system of equations.

$3x-2y+z=1$ (1)
$x+y-2z=5$ (2)
$2x+y+z=0$ (3)

Solution:

$-3\times(2)$: $-3x-3y+6z=-15$ (4)
$3x-2y+z=1$ (1)
(4)+(1): $-5y+7z=-14$ (5)

$2\times(1)$: $6x-4y+2z=2$ (6)
$-3\times(3)$: $-6x-3y-3z=0$ (7)
(6)+(7): $-7y-z=2$ (8)

$7\times(8)$: $-49y-7z=14$ (9)
$-5y+7z=-14$ (5)
(9)+(5): $-54y=0$
$y=0$

Substitute $y=0$ in (8).

$-7y-z=2$
$-7(0)-z=2$
$z=-2$

Substitute $y=0$ and $z=-2$ in (1).

$$\begin{aligned} 3x-2y+z&=1\\ 3x-2(0)-2&=1\\ 3x&=3\\ x&=1 \end{aligned}$$

The solution is $x=1$, $y=0$, and $z=-2$.

Since $x=1$, $y=0$, and $z=-2$ defines the point $(1,0,-2)$ in space, then this point is unique and distinct. The system of equations is independent.

You should also observe that the three equations represent the planes that intersect at one point.

There can also be a system of three equations in three variables such that the equations do not intersect at a single point in space.

Study the following example.

Example 3

Find the solution set of the following system.

$$\begin{aligned} x-y+2z&=1 \quad (1)\\ 3x-y+9z&=-2 \quad (2)\\ x+y+5z&=4 \quad (3) \end{aligned}$$

Solution:

$$\begin{array}{rrl} & 3x-y+9z=-2 & (2)\\ & x-y+2z=1 & (1)\\ \hline (2)-(1): & 2x+7z=-3 & (4) \end{array}$$

$$\begin{array}{rrl} & x-y+2z=1 & (1)\\ & x+y+5z=4 & (3)\\ \hline (1)+(3): & 2x+7z=5 & (5) \end{array}$$

Solve (4) and (5) for x and z.

$$\begin{array}{rrl} & 2x+7z=-3 & (4)\\ & 2x+7z=5 & (5)\\ \hline (4)-(5): & 0x+0z=-8 & (6) \end{array}$$

This is not possible. In other words, there are no values for x, y, and z that will satisfy $0x+0y=0z=-8$. (Note that equations (1), (5), and (6) form an equivalent reduced system.) Therefore, the solution in the empty set and the system is said to be inconsistent.

The preceding system of equations represents three parallel planes in space.

When the system of equations has three variables and there are an infinite number of solutions, then the system is dependent.

Look at the following example which shows a dependent system.

Example 4

Solve the system of equations.

$2x+4y-z=3$ (1)
$x-y+2z=1$ (2)
$x+5y-3z=2$ (3)

Solution:

$-2\times(2)$: $-2x+2y-4z=-2$ (4)
$2x+4y-z=3$ (1)
(4)+(1): $6y-5z=1$ (5)

$-1\times(2)$: $-x+y-2z=-1$ (6)
$x+5y-3z=2$ (3)
(6)+(3): $6y-5z=1$ (7)

Solve for y or z from (5) and (7).

$6y-5z=1$ (5)
$6y-5z=1$ (7)
(5)−(7): $0y+0z=0$ (8)

Equations (1), (5), and (8) are an equivalent reduced system.
Equation (8) is satisfied by all values of z.

Let z be equal to some real number k.

Substitute $z=k$ in (5).

$$6y-5k=+1$$
$$6y=5k+1$$
$$y=\frac{5k+1}{6}$$

Substitute for y and z in (1).

$$2x+4\left(\frac{5k+1}{6}\right)-k=3$$
$$12x+4(5k+1)-6k=18$$
$$12x+20k+4-6k=18$$
$$12x=14-14k$$
$$x=\frac{7-7k}{6}$$

Therefore, in terms of k the solution is $\left(\frac{-7k+7}{6}, \frac{5k+1}{6}, k\right)$ where k is any real number.

You can find any specific solution by substituting various values for k.

For example, if $k = 0$, the solution would be $\left(\frac{7}{6}, \frac{1}{6}, 0\right)$; if $k = 1$, the solution would be $(0, 1, 1)$; and so forth.

Clearly the system of equations with three variables is dependent if there is an infinite number of solutions. When this is the case, the system will generate an infinite set of points forming a line which is the line of intersection of each plane. You should note here that if the equations of a system is mainly a multiple of one equation, the solution will be one distinct plane.

For example, the solution for the following system of equations will define a single plane.

$$\left.\begin{aligned} x+y-2z&=3\\ 2x+2y-4z&=6\\ -3x-3y+6z&=-9 \end{aligned}\right\}$$

Homogeneous Equations

All the equations that you have worked with so far are considered nonhomogeneous because all the equations have contained nonzero constant terms. A system of linear equations is **homogeneous** if all the constant terms are zero.

A linear homogeneous equation in x and y is of the form $Ax + By = 0$; and in x, y, and z, it is of the form $Ax + By + Cz = 0$. A linear homogeneous equation contains only variables of the first power and a constant term of zero.

The equations $4x + 3y = 20$ and $4x + 3y - 2z = 10$ are both linear nonhomogeneous equations since their constant terms are not zero.

Homogeneous equations are never inconsistent because $(0, 0)$ or $(0, 0, 0)$ is always a solution.

Here are three systems of linear homogeneous equations.

$$\left.\begin{aligned} x-5y&=0\\ 7x+2y&=0 \end{aligned}\right\} \qquad \left.\begin{aligned} 3x+y-5z&=0\\ 4x-2y+3z&=0 \end{aligned}\right\} \qquad \left.\begin{aligned} 2x+y-3z&=0\\ 4x-y+7z&=0\\ x+3y-2z&=0 \end{aligned}\right\}$$

The first system will have at least one solution, namely $(0,0)$. The last two systems will have $(0,0,0)$ as a solution.

Given such systems of homogeneous equations, you need to be able to determine whether they are dependent or independent.

Since $(0,0)$ or $(0,0,0)$ is always a solution, the graph of a homogeneous equation always passes through the origin.

You should try a few examples to be able to distinguish between dependent and independent systems of equations.

Example 5

Solve the following homogeneous system of equations.

$$x-y+3z=0 \quad (1)$$
$$3x-y-2z=0 \quad (2)$$
$$5x+2y-3z=0 \quad (3)$$

Solution:

$$\begin{array}{rrl} & 3x-y-2z=0 & (2) \\ & x-y+3z=0 & (1) \\ (2)-(1): & 2x-5z=0 & (4) \end{array}$$

$$\begin{array}{rrl} 2\times(2): & 6x-2y-4z=0 & (5) \\ & 5x+2y-3z=0 & (3) \\ (5)+(3): & 11x-7z=0 & (6) \end{array}$$

$$\begin{array}{rrl} 11\times(4): & 22x-55z=0 & (7) \\ -2\times(6): & -22x+14z=0 & (8) \\ (7)+(8): & -41z=0 & (9) \end{array}$$

Thus, the following equations are an equivalent reduced system.

$$\left.\begin{array}{r} x-y+3z=0 \\ 2x-5z=0 \\ -41z=0 \end{array}\right\}$$

From the equivalent reduced system you will see that the only solution is $(0,0,0)$. Therefore, the original system is independent.

Example 6

Solve the following system of equations.

$$x-5y+6z=0 \quad (1)$$
$$2x-10y+12z=0 \quad (2)$$
$$-3x+15y-18z=0 \quad (3)$$

Solution:

$$\begin{array}{rrl} & 2x-10y+12z=0 & (2) \\ 2\times(1): & 2x-10y+12z=0 & (4) \\ (2)+(4): & 0x+0y+0z=0 & (5) \end{array}$$

$x-5y+6z=0$ (1)
$0y+0z=0$ (6)
$0z=0$ (7)

Since $0x=0$ and $0y=0$, then you can derive these last two equations from (5).

This is an equivalent reduced system of equations.

From (7) you see that z is an arbitrary real number which will be called l.

From (6) you see that y is also an arbitrary number which will be called k.

Substitute $y=k$ and $z=l$ in (1).

$x-5k+6l=0$

$\therefore x=5k-6l$

The solution is $(5k-6l, k, l)$ where l and k are real numbers.

Since there are an infinite number of solutions, then the system is dependent.

You are probably wondering what will happen if there are fewer equations in the system than there are variables. It simply means that you won't have enough information to determine the exact value of each variable. There will be an infinite number of possible solutions for the system. When this is the case you have a dependent system.

Study the following example.

Example 7

Solve the system of homogeneous equations.

$x+y-5z=0$ (1)
$2x+y-z=0$ (2)

Solution:

$-2\times$(1): $-2x-2y+10z=0$ (3)
$2x+y-z=0$ (2)
(3)+(2): $0x-y+9z=0$ (4)

Therefore, equations (1) and (4) are an equivalent reduced system.

In (4), z can be any real number k; therefore, you can solve for y.

$-y+9k=0$

$y=9k$

Substitute $y=9k$ and $z=k$ in (1).

$x+y-5z=0$

$x+9k-5k=0$

$x=-4k$

The solution for the system is $(-4k, 9k, k)$ where k is a real number. Therefore, this system is dependent.

Now try the following problems. (Do any four.)

1. Solve the system of linear equations.

$x+y=10$ (1)
$y+z=8$ (2)
$x+z=12$ (3)

2. Find the value of the variables in the system of equations. Verify your answer.

$5+4z=y+3$ (1)
$x+y=3z$ (2)
$7z=x+2$ (3)

3. Show that the following system of equations is independent.

$x+y=14$ (1)
$y+z=9$ (2)
$z+x=11$ (3)

4. Sketch the plane that is determined by the equation $4x+2y+z=8$.

5. Examine the following system of equations, and classify the system as dependent, independent, or inconsistent.

$x+y+z=1$ (1)
$2x+3y+z=0$ (2)
$x-y+2z=4$ (3)

6. Examine the following systems of homogeneous equations. Classify them as independent or dependent. State the solution set for each.

a. $x+2y=0$ (1)
$4x-y=0$ (2)

b. $x-y+2z=0$ (1)
$3x+2y-5z=0$ (2)

c. $x+2y-z=0$ (1)
$2x+4y-2z=0$ (2)
$-3x-6y+3z=0$ (3)

For solutions to **Activity 1**, turn to **Appendix A**, **Topic 2**.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

} You may decide to do both.

Extra Help

In this topic you have done some work with dependent systems, inconsistent systems, and independent systems. One important task was always to write systems of equations in the form of an equivalent reduced system.

Here is a chance for you to review the techniques in reducing some of these systems.

Example 8

Find an equivalent reduced system for the following system.

$x+3y=7$ ①
$x+3y=-2$ ②

Solution:

Since x and y will be the same value respectively in ① and ②, it is quite obvious that there are no unique solutions for the system.

However, the technique to follow is this:

$$\begin{array}{rl} x+3y=7 & ① \\ x+3y=-2 & ② \\ \hline 0x+0y=9 & \end{array}$$

① − ②:

The reason you write $0x$ and $0y$ is to show the place of the variables.

Therefore, either of the following may be considered equivalent reduced systems.

$$\left.\begin{array}{r} x+3y=-2 \\ 0x+0y=9 \end{array}\right\} \qquad \left.\begin{array}{r} x+3y=7 \\ 0x+0y=9 \end{array}\right\}$$

The fact that $0+0=9$ is already inconsistent tells you that a system with $0x+0y=9$ must also be inconsistent.

If the reduced system is equivalent to the original system, then the original system must also be inconsistent.

Try another example.

Example 9

Find an equivalent reduced system for the following system of equations.

$$x+2y-z=5 \quad (1)$$
$$3x+9y-z=8 \quad (2)$$
$$2x+10y+2z=-3 \quad (3)$$

Solution:

$$\begin{array}{rll} & 3x+9y-z=8 & (2) \\ 3\times(1): & 3x+6y-3z=15 & (4) \\ \hline (2)-(4): & 0x+3y+2z=-7 & (5) \end{array}$$

$$\begin{array}{rll} & 2x+10y+2z=-3 & (3) \\ 2\times(1): & 2x+4y-2z=10 & (6) \\ \hline (3)-(6): & 0x+6y+4z=-13 & (7) \end{array}$$

$$\begin{array}{rll} & 0x+6y+4z=-13 & (7) \\ 2\times(5): & 0x+6y+4z=-14 & (8) \\ \hline (7)-(8): & 0x+0y+0z=1 & (9) \end{array}$$

Thus, equations (1), (5), and (9) are equivalent reduced system.

There are no values of x, y, and z that will satisfy (9). Therefore, the system has no solution. The system is inconsistent.

Try the following questions.

Replace each of the following systems of equations by an equivalent reduced system, and give the solution to the system. Determine if the system is inconsistent, independent, or dependent.

1. $\left.\begin{array}{r} 5x+4y=8 \\ 2x-y=11 \end{array}\right\}$

2. $\left.\begin{array}{r} 8x-4y=5 \\ 4x-2y=8 \end{array}\right\}$

3. $\left.\begin{array}{r} x-y=-3 \\ x+2y=12 \end{array}\right\}$

4. $\left.\begin{array}{r} x-2y+4z=3 \\ 2x+y-2z=5 \\ 4x-3y+6z=7 \end{array}\right\}$

5. $\left.\begin{array}{r} x+2y-z=3 \\ 6x-2y+5z=6 \\ 2x+y+z=4 \end{array}\right\}$

6. $\left.\begin{array}{r} x+2y-z=3 \\ 6x-y+z=4 \\ 4x-5y+3z=-2 \end{array}\right\}$

For solutions to **Extra Help**, turn to **Appendix A, Topic 2**.

Extensions

As you study systems of equations you probably wonder about the significance of them and what practical purpose they serve.

The answer to this question may vary from one person to another. By doing these types of exercises, you organize yourself and you get to appreciate sequential reasoning. Also, it is an excellent tool for problem solving. Here is an example you can consider.

Example 10

A dealer wishes to blend two grades of hickory in order to arrive at a moderate mix for his retailers. Hickory worth \$7.50/kg is blended with hickory worth \$9.50/kg to make 100 kg of the moderate mix to be sold at \$8.60/kg. How much of each brand of hickory should the dealer use?

Solution:

Let x be the \$7.50/kg brand of hickory and y be the \$9.50/kg brand. The value of x kg of the first brand is \$7.5 x, and the value of y kg of the other brand is \$9.5 y. The value of the blend is $\$(7.5x+9.5y)$. This is equal to $100\times\$8.60=\$860$. You can now frame your equations as shown.

$$7.5x+9.5y=860 \quad (1)$$
$$x+y=100 \quad (2)$$

$$2\times(1): \quad 15x+19y=1720 \quad (3)$$
$$-15\times(2): \quad -15x-15y=-1500 \quad (4)$$
$$(3)+(4): \quad 4y=220$$
$$y=55$$

Substitute $y=55$ in (3).

$$x+55=100$$
$$x=45$$

Therefore, the dealer blended 45 kg of the \$7.50/kg hickory with 55 kg of the \$9.50/kg hickory to acquire the required strength.

More often you meet problem solving types that require you to relate numbers to each other to form equations. Then the equations need to be solved in order to calculate the numbers. Here is an example that involves three variables.

Example 11

The sum of three numbers is 2. Three times the first number minus two times the second is equal to 7. Three times the third number plus two times the second is equal to 5. Find the three numbers.

Solution:

Let the three numbers be x, y, and z respectively.

$$x+y+z=2 \quad (1)$$
$$3x-2y=7 \quad (2)$$
$$3z+2y=5 \quad (3)$$

$$\begin{array}{rl} & 3x-2y=7 \quad (2) \\ & 3z+2y=5 \quad (3) \\ (2)+(3): & 3x+3z=12 \quad (4) \end{array}$$

$$\begin{array}{rl} 2\times(1): & 2x+2y+2z=4 \quad (5) \\ & 3x-2y=7 \quad (2) \\ (5)+(2): & 5x+2z=11 \quad (6) \end{array}$$

$$\begin{array}{rl} -2\times(4): & -6x-6z=-24 \quad (7) \\ 3\times(6): & 15x+6z=33 \quad (8) \\ (7)+(8): & 9x=9 \\ & x=1 \end{array}$$

Substitute $x=1$ in (2).

$$\begin{aligned} 3(1)-2y&=7 \\ -2y&=7-3 \\ y&=-2 \end{aligned}$$

Substitute $y=-2$ in (3).

$$\begin{aligned} 3z+2(-2)&=5 \\ 3z&=5+4 \\ z&=3 \end{aligned}$$

The three numbers are 1, −2, and 3.

Now do any four of the following problems.

1. If the numerator and denominator of a fraction are each increased by 4, the fraction will be equal to $\frac{7}{9}$. If, however, the terms are decreased by 3, then the fraction is equal to $\frac{3}{4}$. Find the fraction.

2. The perimeter of a triangle *ABC* is 30 m. The sum of sides *AB* and *BC* is 17 m; of sides *AB* and *AC* is 18 m; and of sides *BC* and *AC* is 25 m. Calculate the following.

   a. the measure of each side
   b. the area of the triangle

3. Three numbers exist such that their sum is 72. The second is the average of the first and third, and the third is the average of the second and twice the first. Find the three numbers.

4. The sums of three numbers taken in pairs are 30, 32, and 36. Find the three numbers.

5. An alloy is made up of three metals, namely gold, silver, and copper. One part of gold weighs 20 g and costs $75, one part of silver weighs 15 g and costs $65, and one part of copper weighs 24 g and costs $50. If eighteen parts of the alloy weigh 372 g and are sold for $1110, how many parts of each metal are being sold?

For solutions to **Extensions**, turn to **Appendix A, Topic 2**.

Topic 3 Matrices – Definitions and Properties

Introduction

Matrices have now become an integral part of the mathematical background necessary for such diverse fields as electrical engineering, education, chemistry, and sociology as well as for statistics and pure mathematics.

In this topic you will learn the basic features necessary to enable you to make worthwhile use of current texts that deal with this subject. You should also be able to work in several fields which require some knowledge of matrix theory.

What Lies Ahead

Throughout the topic you will learn to

1. define matrices and their related properties
2. solve systems of equations using matrices

Now that you know what to expect, turn the page to begin your study of matrices.

Exploring Topic 3

Activity 1

Define matrices and their related properties.

A matrix is a rectangular array of numbers enclosed in square brackets. The numbers in a matrix are referred to as the **entries**, the **elements**, or the **components** of the matrix. Matrices are used to facilitate the study of problem solving.

The **order** or **dimension** of a matrix is given by stating the number of rows and then the number of columns. For example a 3×4 matrix has three rows and four columns.

If the elements of the matrix are real numbers, it is called a **real matrix**. A **square matrix** has the same number of rows and columns. A matrix containing only one row is called a **row matrix**, while a matrix containing only one column is called a **column matrix**. The entries in a **zero matrix** are all zeros.

Two matrices are **equal** when they have the same dimension and have the same elements in corresponding position.

The following are examples of matrices.

$$\begin{bmatrix} 0 & 3 \\ 1 & 4 \\ 2 & 7 \end{bmatrix}$$

This is a 3×2 matrix containing three rows and two columns.

$$\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$$

This is a 3×1 matrix containing three rows and one column. It is also referred to as a column matrix.

$$\begin{bmatrix} 3 & -7 \end{bmatrix}$$

This is a 1×2 matrix containing one row and two columns. It is also referred to as a row matrix.

$$\begin{bmatrix} 0 & 3 & 2 \\ 1 & 4 & -5 \\ 2 & 7 & 6 \end{bmatrix}$$

This is a 3×3 matrix containing three rows and three columns. This is also a square matrix.

$$\begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$$

This is a 2×2 square matrix.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

When all the entries in a matrix are zeros, the matrix is referred to as a zero matrix. This is a 2×3 zero matrix.

The real numbers in a matrix are generally referred to as elements.

Always state the rows before the columns.

For more useful purposes you will often associate two different matrices with every system of m linear equations in n variables.

For example, examine the following system of equations.

$$\left.\begin{aligned} x-3y&=5\\ 4x+y&=8 \end{aligned}\right\}$$

This system has $\begin{bmatrix} 1 & -3 \\ 4 & 1 \end{bmatrix}$ as the **matrix of coefficients**. The elements are the coefficients of the x and y terms.

If you include the constant term of the system with the coefficients of the variables, the resulting array is called the **augmented matrix** of the system.

The augmented matrix is as follows:

$$\begin{bmatrix} 1 & -3 & 5 \\ 4 & 1 & 8 \end{bmatrix}$$

Study the following example.

Example 1

Here is a system of equations.

$$3x-y+z-6=0 \quad (1)$$
$$4x-2y+3z+5=0 \quad (2)$$

- Write the matrix of coefficients.

  Solution:

  First, write (1) and (2) as follows:

  $$3x-y+z=6 \quad (1)$$
  $$4x-2y+3z=-5 \quad (2)$$

  Therefore, the matrix of coefficients is

  $$\begin{bmatrix} 3 & -1 & 1 \\ 4 & -2 & 3 \end{bmatrix}.$$

- Write the augmented matrix of the system.

  Solution:

  The augmented matrix is

  $$\begin{bmatrix} 3 & -1 & 1 & 6 \\ 4 & -2 & 3 & -5 \end{bmatrix}.$$

The constant term must be to the **right** of the equal sign.

Row-Equivalent Matrices

Two matrices are said to be row equivalent if each can be obtained from the other by a sequence of operations of the following three types.

Type 1: Obtaining a matrix by interchanging two rows indicates row equivalence.

Thus, $\begin{bmatrix} 2 & -7 \\ 3 & 5 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 3 & 5 \\ 2 & -7 \end{bmatrix}$.

You should note here that performing certain operations on the equations of a system will yield equivalent systems. Interchanging the rows of a matrix can be compared to switching the position of the equations in a system.

You should also note that the matrices are row equivalent but not necessarily equal.

Type 2: Obtaining a matrix by multiplying a row by a nonzero real number indicates row equivalence.

Thus, $\begin{bmatrix} 2 & -7 \\ 3 & 5 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 2 & -7 \\ 9 & 15 \end{bmatrix}$ because the second row is multiplied by 3. Again the two matrices are row equivalent but not equal.

Type 3: Obtaining a matrix by adding a multiple of one row to the corresponding elements of another row indicates row equivalence.

For example, take $\begin{bmatrix} 2 & -7 \\ 3 & 5 \end{bmatrix}$. By adding 3 times row 1 to row 2, $\begin{bmatrix} 2 & -7 \\ 3 & 5 \end{bmatrix}$ becomes

$$\begin{bmatrix} 2 & -7 \\ 3(2)+3 & 3(-7)+5 \end{bmatrix} \text{ which is } \begin{bmatrix} 2 & -7 \\ 9 & -16 \end{bmatrix}.$$

The middle step is never shown. This calculation is done mentally with experience.

It is not always easy to look at two matrices and tell if they are row equivalent or not. Who could tell that $\begin{bmatrix} 2 & -7 \\ 3 & 5 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 2 & -7 \\ 9 & -16 \end{bmatrix}$?

This is what you have to do eventually. With practice you will learn to tell at a glance.

Two matrices are equal if and only if one is a duplicate of the other.

One other thing you should observe is that the rows which are unchanged are always retained in the new matrix.

You should also know that there are many ways of forming row-equivalent matrices, so there is more than one correct answer. The solution is rarely unique.

Denote row 1 as ①, row 2 as ②, and so on. These symbols refer to the matrix immediately preceding the one at hand. Put the operation beside the row being altered and list successive row-equivalent matrices vertically.

Here is an example where specific instructions are given.

Example 2

Find the row-equivalent matrix of $\begin{bmatrix} 5 & -1 \\ 2 & 6 \end{bmatrix}$ by successively performing the following operations.

- Multiply row 2 by $\frac{1}{2}$.

Solution:

$$\begin{bmatrix} 5 & -1 \\ 2 & 6 \end{bmatrix} \quad \begin{matrix} ① \\ ② \end{matrix}$$

$$\tfrac{1}{2}\times② : \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}$$

The unchanged rows are always repeated in successive matrices.

- Add -5 times row 2 to row 1.

Solution:

$$-5\times② + ① : \begin{bmatrix} 0 & -16 \\ 1 & 3 \end{bmatrix}$$

These row numbers refer to the preceding matrix. Since -5 times row 2 is being added to row 1, then the change is made to row 1.

- Interchange the rows.

Solution:

$$① \leftrightarrow ② : \begin{bmatrix} 1 & 3 \\ 0 & -16 \end{bmatrix}$$

Therefore, the matrix $\begin{bmatrix} 5 & -1 \\ 2 & 6 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & 3 \\ 0 & -16 \end{bmatrix}$ according to the given instructions.

The steps outlined above for a row-equivalent matrix are similar to those for a column-equivalent matrix.

Two matrices are column equivalent if each can be obtained from the other by any of the following procedures.

- the interchange of any two columns
- the multiplication of a column by any nonzero real number
- to the elements of one column add a multiple of the corresponding elements of another column

When solving a system of equations, try to eliminate some of the unknowns or variables. By doing this you will have a less difficult task in solving for the others. As mentioned previously, a system of equations can be represented by an augmented matrix of the system.

Row-Reduced Echelon Form

Your aim is to be able to reduce a matrix to an equivalent form that looks like this:

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(where * represents any real number)

* = any real number

The first nonzero entry in each row is 1, and each row contains more initial zero terms than the preceding rows. It may look like this:

$$\begin{bmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The first nonzero entry from the left is always 1 in any row. The first 1 is always to the right of that in the preceding row although not necessarily along the diagonal.

The element in the first row and first column must be 1. Also, each row contains more initial zero terms than the preceding rows. An equivalent matrix of this form is called a **row-reduced echelon form** of the augmented matrix.

The solution of a system of equations can be read from the row-reduced echelon matrix.

The object now is to find a row-reduced echelon form of a given matrix. The first step is to get a 1 in the upper left position. Now you can use this 1 to arrive at initial zeros in all the other rows. Next, you want a 1 in the second position of the row 2. You can use this 1 to arrive at second position zeros in all of the rows after row 2 and so on.

Observe the following method for identifying the location of each element in a matrix.

$$\begin{array}{l} \text{row 1, column 1} \longrightarrow \\ \text{row 2, column 1} \longrightarrow \end{array} \begin{bmatrix} 5 & 8 \\ 4 & 3 \end{bmatrix} \begin{array}{l} \longleftarrow \text{row 1, column 2} \\ \longleftarrow \text{row 2, column 2} \end{array}$$

The element 5 lies in row 1 and column 1.
The element 4 lies in row 2 and column 1.
The element 8 lies in row 1 and column 2.
The element 3 lies in row 2 and column 2.

The following examples show the procedures in obtaining row-reduced echelon forms for matrices.

Example 3

Find a row-reduced echelon form for the matrix $\begin{bmatrix} 5 & 8 \\ 4 & 3 \end{bmatrix}$.

Solution:

Since the element in the first row, first column must be 1, you can multiply row 1 by $\frac{1}{5}$.

$\frac{1}{5}\times(1): \begin{bmatrix} 1 & \frac{8}{5} \\ 4 & 3 \end{bmatrix}$ An operation was performed on row 1 in the preceding matrix to give this step.

Since you want a zero (instead of 4) at row 2, column 1, you multiply (1) by -4 and add to row 2.

$-4\times(1)+(2): \begin{bmatrix} 1 & \frac{8}{5} \\ 0 & -\frac{17}{5} \end{bmatrix}$ The operation performed on element 3 in row 2 of the preceding matrix was $3+-4\left(\frac{8}{5}\right)=3-\frac{32}{5}=-\frac{17}{5}$.

Since the first nonzero element in each row must be 1, you must change $-\frac{17}{5}$ to 1. In order to do this you use the multiplicative inverse of $-\frac{17}{5}$.

$-\frac{5}{17}\times(2): \begin{bmatrix} 1 & \frac{8}{5} \\ 0 & 1 \end{bmatrix}$ One operation was performed on row 2 in the preceding matrix to arrive at the row-reduced echelon form.

Thus, $\begin{bmatrix} 1 & \frac{8}{5} \\ 0 & 1 \end{bmatrix}$ is a row-reduced echelon form of $\begin{bmatrix} 5 & 8 \\ 4 & 3 \end{bmatrix}$.

The same problem may be solved using different steps to arrive at a row-reduced echelon form which is different from the last. You should realize that they are row-equivalent matrices even though they look different.

Example 4

Find a row-reduced echelon form for the matrix $\begin{bmatrix} 5 & 8 \\ 4 & 3 \end{bmatrix}$.

Solution:

$$\begin{bmatrix} 5 & 8 \\ 4 & 3 \end{bmatrix} \begin{matrix} ① \\ ② \end{matrix}$$

$-1\times② + ①$: $\begin{bmatrix} 1 & 5 \\ 4 & 3 \end{bmatrix}$

$-4\times① + ②$: $\begin{bmatrix} 1 & 5 \\ 0 & -17 \end{bmatrix}$

$-\frac{1}{17}\times②$: $\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$

The matrix $\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$ is a row-reduced echelon form of $\begin{bmatrix} 5 & 8 \\ 4 & 3 \end{bmatrix}$.

This solution is different from that obtained previously; however, they are row equivalent.

You should try to show that they are row equivalent.

The following example shows a 3×3 matrix being changed to a row-reduced echelon form.

Example 5

Find a row-reduced echelon form for the matrix $\begin{bmatrix} 3 & -1 & 2 \\ 1 & 5 & -3 \\ -2 & 4 & 0 \end{bmatrix}$.

Solution:

$$\begin{bmatrix} 3 & -1 & 2 \\ 1 & 5 & -3 \\ -2 & 4 & 0 \end{bmatrix} \begin{matrix} ① \\ ② \\ ③ \end{matrix}$$

$① \leftrightarrow ②$: $\begin{bmatrix} 1 & 5 & -3 \\ 3 & -1 & 2 \\ -2 & 4 & 0 \end{bmatrix}$

Interchange ① and ② to obtain the element 1 located in row 1, column 1.

A zero is required in row 2, column 1.

$$-3\times(1)+(2):\begin{bmatrix}1 & 5 & -3\\ 0 & -16 & 11\\ -2 & 4 & 0\end{bmatrix}$$

Now an element 1 is required in row 2, column 2.

$$\tfrac{-1}{16}\times(2):\begin{bmatrix}1 & 5 & -3\\ 0 & 1 & -\tfrac{11}{16}\\ -2 & 4 & 0\end{bmatrix}$$

Now a zero is required for row 3, column 1.

$$2\times(1)+(3):\begin{bmatrix}1 & 5 & -3\\ 0 & 1 & -\tfrac{11}{16}\\ 0 & 14 & -6\end{bmatrix}$$

Another zero is required for row 3, column 2.

$$-14\times(2)+(3):\begin{bmatrix}1 & 5 & -3\\ 0 & 1 & -\tfrac{11}{16}\\ 0 & 0 & \tfrac{29}{8}\end{bmatrix}$$

An element 1 is needed for row 3, column 3.

$$\tfrac{8}{29}\times(3):\begin{bmatrix}1 & 5 & -3\\ 0 & 1 & -\tfrac{11}{16}\\ 0 & 0 & 1\end{bmatrix}$$

This is a row-reduced echelon form of the given matrix. (Note that this solution is not unique.)

$$(-3)(5)+(-1)=-16$$
$$(-3)(-3)+2=11$$

$$(-14)\left(\frac{-11}{16}\right)+(-6)=\frac{154}{16}-6$$
$$=\frac{154-96}{16}$$
$$=\frac{58}{16}$$
$$=\frac{29}{8}$$

Do any five of the following problems.

1. Write out the matrix of coefficients and the augmented matrix for each of the following equation systems.

   a. $\left.\begin{aligned} x+y&=8\\ 2x+y&=7 \end{aligned}\right\}$

   b. $\left.\begin{aligned} 4x-5y&=-3\\ 2x+7y&=14 \end{aligned}\right\}$

   c. $\left.\begin{aligned} x+2y+z&=0\\ 2x-3y-z&=6\\ 3x+5y+4z&=5 \end{aligned}\right\}$

2. Successively perform the following operations on the given matrix. Use ① and ② to indicate row 1 and row 2.

$$\begin{bmatrix} 3 & -1 & -7 \\ 5 & 2 & 4 \end{bmatrix} \begin{matrix} ① \\ ② \end{matrix}$$

   a. Interchange the two rows.
   b. Add 4 times ① to ②.
   c. Multiply ① by 6.

3. Successively perform the following operations on the given matrix. Use ① and ② to indicate row 1 and row 2.

$$\begin{bmatrix} 2 & -7 \\ 5 & 0 \end{bmatrix} \begin{matrix} ① \\ ② \end{matrix}$$

   a. Add 3 times ② to ①.
   b. Interchange the rows.
   c. Multiply ② by 3.

4. Successively perform the following operations on the given matrix.

$$\begin{bmatrix} 3 & -1 & 2 \\ 5 & 1 & 2 \\ 0 & 1 & 5 \end{bmatrix} \begin{matrix} ① \\ ② \\ ③ \end{matrix}$$

   a. Interchange ① and ③.
   b. Multiply ② by 2.
   c. Add 3 times ② to ③.
   d. Interchange ① and ②.

5. In the blanks beside the matrices, explain the operations in obtaining the row-equivalent matrix.

$$\begin{bmatrix} 3 & -2 & 7 \\ 1 & 4 & -2 \\ 2 & -1 & 3 \end{bmatrix} \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

_____ $\begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 2 & -1 & 3 \end{bmatrix}$

_____ $\begin{bmatrix} 1 & -1 & 4 \\ 2 & -1 & 3 \\ 1 & 4 & -2 \end{bmatrix}$

_____ $\begin{bmatrix} 1 & -1 & 4 \\ 8 & -4 & 12 \\ 1 & 4 & -2 \end{bmatrix}$

_____ $\begin{bmatrix} 1 & -1 & 4 \\ \frac{17}{2} & \frac{-9}{2} & 14 \\ 1 & 4 & -2 \end{bmatrix}$

6. Find a row-reduced echelon form for the following matrix.

$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 4 & 6 \end{bmatrix}$$

7. Find a row-reduced echelon form for the following matrix.

$$\begin{bmatrix} 3 & -1 & 2 \\ -1 & 4 & 9 \end{bmatrix}$$

8. Find a row-reduced echelon form for the following matrix.

$$\begin{bmatrix} 2 & 4 & 6 \\ 5 & 2 & 3 \end{bmatrix}$$

9. Find the row-reduced echelon form for the augmented matrix of the following system of equations.

$$\left.\begin{aligned} x+4y&=-3 \\ 3x-2y&=1 \\ 2x-y&=3 \end{aligned}\right\}$$

For solutions to **Activity 1**, turn to **Appendix A**, **Topic 3**.

Activity 2

Solve systems of equations using matrices.

For most students it presents quite a task to express a given matrix in row-reduced echelon form. You might wonder why you have to endure this in the study of matrices. Well, one reason is to use this knowledge to minimize your work when you seek to solve systems of equations.

Once you have your row-reduced echelon form, you can simply read the solutions to the equations. You can also tell if the system is dependent, independent, or inconsistent.

Now it will be shown that a row-reduced echelon form of the augmented matrix of a system of equations is the augmented matrix of an equivalent reduced system of equations.

In the next example you will find the row-reduced echelon form of an augmented matrix of a system of equations and compare it with the operations used to solve that same system of equations.

Example 6

Examine the following system of equations.

$3x-2y=7$ (1)
$5x+3y=8$ (2)

- Write the augmented matrix.

Solution:

The augmented matrix is $\begin{bmatrix} 3 & -2 & 7 \\ 5 & 3 & 8 \end{bmatrix}$.

- Change the augmented matrix to row-reduced echelon form.

Solution:

$\begin{bmatrix} 3 & -2 & 7 \\ 5 & 3 & 8 \end{bmatrix}$ (1) (2) Note that row (1) here coincides with equation (1) from the original system, and likewise for row (2).

$3\times(2)$: $\begin{bmatrix} 3 & -2 & 7 \\ 15 & 9 & 24 \end{bmatrix}$

$-5\times(1)+(2)$: $\begin{bmatrix} 3 & -2 & 7 \\ 0 & 19 & -11 \end{bmatrix}$

$\frac{1}{3}\times(1)$: $\begin{bmatrix} 1 & \frac{-2}{3} & \frac{7}{3} \\ 0 & 1 & \frac{-11}{19} \end{bmatrix}$

$\frac{1}{19}\times(2)$:

- Obtain an equivalent reduced system of equations by applying the operations used previously.

Solution:

$3x-2y=7$ (1)
$5x+3y=8$ (2)

$3\times(2)$: $15x+9y=24$ (3)
$-5\times(1)$: $-15x+10y=-35$ (4)
(3)+(4): $0x+19y=-11$ (5)

$y=\frac{-11}{19}$ (6)

$\frac{1}{3}\times(1)$: $x-\frac{2}{3}y=\frac{7}{3}$ (7)

Notice that (6) and (7) are the equivalent reduced system of equations and a row-reduced echelon form of the augmented matrix. Now the solution of the original system can be obtained.

Notice that when an augmented matrix of a system of equations is reduced to row-reduced echelon form, the same procedures will change the system to an equivalent reduced system.

Study the following demonstrations that show how the row-reduced echelon form can be related to systems of equations that are in reduced equivalent form. It is also possible to determine if the system is dependent, independent, or inconsistent.

Demonstration 1

$\begin{bmatrix}1 & 4 & -2\\ 0 & 1 & 4\end{bmatrix}$ corresponds to $x+4y=-2$ and $y=4$.

The matrix indicates that one of the roots of the system is 4 and that the other root is simply calculated from the other equation.

When $y=4$, the following may be stated:

$$x+4(4)=-2$$
$$x=-16-2$$
$$=-18$$

Therefore, the solution is $x=-18$ and $y=4$ or $(-18,4)$.

Note: If you substitute $y=\frac{-11}{19}$ in (6), then $x=\frac{7}{3}+\frac{2}{3}\left(\frac{-11}{19}\right)=\frac{111}{57}$ $=1\frac{18}{19}$.

Since there is a unique solution, then the original system is independent.

Demonstration 2

$\begin{bmatrix} 1 & 4 & -3 \\ 0 & 1 & 0 \end{bmatrix}$ corresponds to $x+4y=-3$ and $y=0$.

The solution for this system is $y=0$ and $x=-3$, since $x+4(0)=-3$.

The solution is $(-3, 0)$; thus, the original system is independent.

Demonstration 3

$\begin{bmatrix} 1 & 3 & 7 \\ 0 & 0 & 1 \end{bmatrix}$ corresponds to $x+3y=7$ and $0x+0y=1$.

Since x and y are each undefined in the row-reduced equivalent system, the solution to the original system is ϕ (the empty set).

Therefore, the original system of equations is inconsistent.

Demonstration 4

$\begin{bmatrix} 1 & 5 & -7 \\ 0 & 0 & 0 \end{bmatrix}$ corresponds to $x+5y=-7$ and $0x+0y=0$.

The second equation is satisfied by all values of x and y. Thus, for the system of solutions, let $y=k$ and then solve for x.

$$\therefore x+5k=-7$$
$$x=-5k-7$$

The general solution would be $x=-5k-7$ and $y=k$, or $[-(5k+7),\ k]$ where k is any real number. Since there are an infinite number of solutions, the original system of equations is dependent.

Try an example that has three variables.

Example 7

Determine a row-reduced echelon form of the augmented matrix of a system. Use the following augmented matrix to obtain the solution of the system.

$$\begin{bmatrix} 1 & 3 & -2 & -1 \\ 3 & -1 & 4 & 7 \\ 1 & 5 & -4 & -3 \end{bmatrix}$$

Solution:

$$-3\times(3)+(2): \begin{bmatrix} 1 & 3 & -2 & -1 \\ 0 & -16 & 16 & 16 \\ 1 & 5 & -4 & -3 \end{bmatrix}$$

$$\tfrac{-1}{16}\times(2):\begin{bmatrix}1 & 3 & -2 & -1\\ 0 & 1 & -1 & -1\\ 1 & 5 & -4 & -3\end{bmatrix}$$

$$-1\times(1)+(3):\begin{bmatrix}1 & 3 & -2 & -1\\ 0 & 1 & -1 & -1\\ 0 & 2 & -2 & -2\end{bmatrix}$$

$$-2\times(2)+(3):\begin{bmatrix}1 & 3 & -2 & -1\\ 0 & 1 & -1 & -1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

The row-reduced echelon form of the augmented matrix corresponds to the following system of equations.

$$\left.\begin{aligned} x+3y-2z&=-1\\ y-z&=-1\\ 0z&=0\end{aligned}\right\}$$

Therefore, if z equals a real number k, then $y=k-1$.

$$\begin{aligned} x&=-3(k-1)+2k-1\\ &=-3k+3+2k-1\\ &=-k+2\end{aligned}$$

The solution is $x=-(k-2)$, $y=k-1$, and $z=k$, or $[-(k-2),\ (k-1),\ k]$ where k is a real number.

Since there are an infinite number of solutions, the system is dependent.

In a dependent system the matrix of the row-reduced echelon form contains all zeros in the bottom row.

It is not easy to look at a system of equations and tell whether or not it is dependent, independent, or inconsistent. However, if you can write the system in its row-reduced echelon form, then you can probably predict the type of system. See if you can predict the type of system in the next example.

Example 8

Write out the augmented matrix for the following system of equations. Change the matrix to row-reduced echelon form, solve for the unknowns, and classify the system a dependent, independent, or inconsistent.

$$\begin{aligned} x-y-2z&=7 \quad (1)\\ 3x+2y-z&=-3 \quad (2)\\ x-2y+z&=5 \quad (3)\end{aligned}$$

Solution:

The augmented matrix for the system of equations is as follows:

$$\begin{bmatrix}1 & -1 & -2 & 7\\ 3 & 2 & -1 & -3\\ 1 & -2 & 1 & 5\end{bmatrix}\begin{matrix}(1)\\(2)\\(3)\end{matrix}$$

(Note that each row of the matrix corresponds to an equation in the system having the same number.)

$$3\times(3)-(2):\begin{bmatrix}1 & -1 & -2 & 7\\ 0 & -8 & 4 & 18\\ 1 & -2 & 1 & 5\end{bmatrix}$$

$$\begin{matrix} \\ \textcircled{1}-\textcircled{3}: \end{matrix}\begin{bmatrix} 1 & -1 & -2 & 7 \\ 0 & -8 & 4 & 18 \\ 0 & 1 & -3 & 2 \end{bmatrix}$$

$$\begin{matrix} \\ -\frac{1}{8}\times\textcircled{2}: \end{matrix}\begin{bmatrix} 1 & -1 & -2 & 7 \\ 0 & 1 & -\frac{1}{2} & -\frac{9}{4} \\ 0 & 1 & -3 & 2 \end{bmatrix}$$

$$\begin{matrix} \\ -1\times\textcircled{2}+\textcircled{3}: \end{matrix}\begin{bmatrix} 1 & -1 & -2 & 7 \\ 0 & 1 & -\frac{1}{2} & -\frac{9}{4} \\ 0 & 0 & -\frac{5}{2} & \frac{17}{4} \end{bmatrix}$$

$$\begin{matrix} \\ -\frac{2}{5}\times\textcircled{3}: \end{matrix}\begin{bmatrix} 1 & -1 & -2 & 7 \\ 0 & 1 & -\frac{1}{2} & -\frac{9}{4} \\ 0 & 0 & 1 & -\frac{17}{10} \end{bmatrix}$$

From the matrix you may state the following:

$$z=-\frac{17}{10}$$

$$y-\frac{1}{2}z=-\frac{9}{4}$$

$$y-\frac{1}{2}\left(-\frac{17}{10}\right)=-\frac{9}{4}$$

$$y=-\frac{31}{10}$$

$$x-y-2z=7$$

$$x-\left(-\frac{31}{10}\right)-2\left(-\frac{17}{10}\right)=7$$

$$x=\frac{1}{2}$$

The solution is $x=\frac{1}{2}$, $y=-\frac{31}{10}$, and $z=-\frac{17}{10}$. Since the solution is unique and distinct, the original system is independent.

Example 9

Examine the following system of equations.

$$x-2y+z=3 \quad \textcircled{1}$$
$$2x+3y-z=4 \quad \textcircled{2}$$
$$4x-y+z=12 \quad \textcircled{3}$$

Determine a row-reduced echelon form of the augmented matrix of the given system and use this matrix to obtain the solution of the system of equations.

Solution:

The augmented matrix is as follows:

$$\begin{bmatrix} 1 & -2 & 1 & 3 \\ 2 & 3 & -1 & 4 \\ 4 & -1 & 1 & 12 \end{bmatrix}\begin{matrix} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \end{matrix}$$

$$\begin{matrix} \\ -2\times\textcircled{1}+\textcircled{2}: \end{matrix}\begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 7 & -3 & -2 \\ 4 & -1 & 1 & 12 \end{bmatrix}$$

$$-4\times(1)+(3):\begin{bmatrix}1 & -2 & 1 & 3\\ 0 & 7 & -3 & -2\\ 0 & 7 & -3 & 0\end{bmatrix}$$

$$-1\times(2)+(3):\begin{bmatrix}1 & -2 & 1 & 3\\ 0 & 7 & -3 & -2\\ 0 & 0 & 0 & 2\end{bmatrix}$$

$$\tfrac{1}{7}\times(2):\begin{bmatrix}1 & -2 & 1 & 3\\ 0 & 1 & -\frac{3}{7} & -\frac{2}{7}\\ 0 & 0 & 0 & 2\end{bmatrix}$$

$$\tfrac{1}{2}\times(3):\begin{bmatrix}1 & -2 & 1 & 3\\ 0 & 1 & -\frac{3}{7} & -\frac{2}{7}\\ 0 & 0 & 0 & 1\end{bmatrix}$$

This row-reduced echelon form corresponds to the following equivalent reduced system.

$$\begin{aligned} x-2y+z&=3\\ y-\tfrac{3}{7}z&=-\tfrac{2}{7}\\ 0z&=1\end{aligned}$$

From $0z=1$ you can tell that the original system has no solution, or in other words the solution is ϕ. Therefore, you can say the original system is inconsistent.

When $[0 \quad 0 \quad 0 \quad 1]$ occurs in the bottom row of the row-reduced echelon form, this indicates that the original system is inconsistent.

As a summary you should be able to draw a conclusion regarding the three types of systems of equations. Examine the following row-reduced echelon forms of matrices.

$$\begin{bmatrix}1 & * & * & *\\ 0 & 1 & * & *\\ 0 & 0 & 1 & *\end{bmatrix}$$

Here, the original system of equations is independent.

$$\begin{bmatrix}1 & * & * & *\\ 0 & 1 & * & *\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Here, the original system of equations is dependent.

$$\begin{bmatrix}1 & * & * & *\\ 0 & 1 & * & *\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Here, the original system of equations is inconsistent.

Now do the following practice exercise.

Do the following questions.

1. Write the system of equations corresponding to each of the following matrices, state its solution set, and tell whether the system is independent, dependent, or inconsistent.

   a. $\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 5 \end{bmatrix}$

   b. $\begin{bmatrix} 1 & 5 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

   c. $\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

   d. $\begin{bmatrix} 1 & 5 & 7 & -7 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

   e. $\begin{bmatrix} 1 & 2 & 5 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

   f. $\begin{bmatrix} 1 & 2 & 5 & 3 \\ 0 & 1 & 1 & 4 \end{bmatrix}$

2. Solve using the row-reduced echelon method, and state whether the system is independent, dependent, or inconsistent.

$$\left.\begin{aligned} x-2y+3z&=4 \\ 3x+y+3z&=5 \\ 2x+4y+z&=2 \end{aligned}\right\}$$

3. If $\begin{bmatrix} 1 & 5 \\ 0 & -17 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 5 & 8 \\ 4 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & \frac{8}{5} \\ 0 & 1 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 5 & 8 \\ 4 & 3 \end{bmatrix}$, then show that $\begin{bmatrix} 1 & \frac{8}{5} \\ 0 & 1 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & 5 \\ 0 & -17 \end{bmatrix}$.

For solutions to **Activity 2**, turn to **Appendix A, Topic 3**.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

} You may decide to do both.

Extra Help

A display of real numbers in rows and columns is called an array. Each real number in the display is called an element or an entry.

A real matrix is a square or rectangular array of real numbers enclosed in square brackets or two solid double lines.

Some texts will write a matrix using double lines.

$$\begin{Vmatrix} 2 & -3 \\ 4 & 0 \end{Vmatrix}$$

In this course you will use the square brackets such as the following:

$$\begin{bmatrix} 2 & -3 \\ 4 & 0 \end{bmatrix}$$

The rows are the elements read in a horizontal manner, and the columns are the elements read in a vertical manner. The two diagonals of a matrix are also of importance. The diagonal from the upper left to the lower right is called the **principal diagonal**, and the diagonal from the lower left to the upper right is called the **secondary diagonal**.

As a rule, always state the number of rows first and then the number of columns. If you speak of an $n \times m$ matrix, you mean there are n rows and m columns.

The following are various matrices of differing dimensions and orientations.

$\begin{bmatrix} 1 & -6 \\ 4 & 3 \end{bmatrix}$ This is a 2×2 square matrix.

$\begin{bmatrix} 1 & \frac{1}{2} & -8 \\ 6 & 0 & 3 \end{bmatrix}$ This is a 2×3 matrix containing two rows and three columns.

$\begin{bmatrix} 0 & 4 & -6 \\ 3 & 1 & 5 \\ 8 & 0 & 9 \end{bmatrix}$ This is a 3×3 square matrix.

$[3]$ This is a 1×1 matrix.

$[4 \quad -7]$ This is a 1×2 matrix and is referred to as a row matrix.

$[2 \quad 0 \quad 6]$ This is a 1×3 matrix and is referred to as a row matrix.

$\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ This is a 2×1 matrix and is referred to as a column matrix.

$\begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix}$ This is a 3×1 matrix and is referred to as a column matrix.

$\begin{bmatrix} 1 & -2 \\ 0 & 3 \\ \frac{1}{2} & 4 \end{bmatrix}$ This is a 3×2 matrix.

One of the main uses of a matrix is to describe or solve systems of equations.

You can form two useful matrices from a system of m linear equations with n variables.

Examine a system of two linear equations with two variables.

$$2x-3y=5$$
$$4x+y=6$$

You can use the coefficients of the variables to form the following matrix.

$$\begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}$$

Another matrix may be formed by the coefficients of the variables and the constant term to the right of the equal sign.

$$\begin{bmatrix} 2 & -3 & 5 \\ 4 & 1 & 6 \end{bmatrix}$$

This 2×3 matrix is called the **augmented matrix**.

Example 10

Write the system of equations in x, y, and z that can be formed from the given augmented matrix.

$$\begin{bmatrix} 2 & 3 & -1 & 4 \\ 1 & -2 & 3 & 5 \\ 0 & 4 & 11 & 8 \end{bmatrix}$$

Solution:

The following system can be formed from the matrix.

$$\left.\begin{aligned} 2x+3y-z&=4 \\ x-2y+3z&=5 \\ 4y+11z&=8 \end{aligned}\right\}$$

In order to solve a system of equations, you can write the augmented matrix, reduce it to a row-reduced echelon matrix, and then solve for the variables of the equations of the system.

There are certain operations that are allowed in changing the augmented matrix to a row-equivalent matrix.

First, you are allowed to interchange any two rows.

For example, $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ may become $\begin{bmatrix} 3 & 4 \\ 2 & -1 \end{bmatrix}$.

By interchanging these two rows you can say the two matrices are row equivalent (but not equal).

Secondly, you are allowed to multiply a row by a nonzero real number.

Thus, $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 2 & -1 \\ 9 & 12 \end{bmatrix}$.

Notice that the second row is multiplied by 3.

Lastly, you are allowed to add a multiple of one row to another row as long as the addition is done to respective elements.

Thus, $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ is equivalent to $\begin{bmatrix} 2 & -1 \\ 9 & 1 \end{bmatrix}$ where 3 times row 1 is added to row 2.

$$3\times\textcircled{1}+\textcircled{2}: \begin{bmatrix} 2 & -1 \\ 3(2)+3 & 3(-1)+4 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 6+3 & -3+4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ 9 & 1 \end{bmatrix}$$

In obtaining the row-equivalent matrix, the intermediate steps are generally not shown. You can do them on scrap paper and record the final steps. However, if they are required, you should show them.

The row(s) that are unchanged are always written again in the new matrix.

If any number of students are seeking a row-equivalent matrix for a single augmented matrix, they can all arrive at a different row-equivalent matrix, yet they could all be correct. In other words, there could be an endless number of row-equivalent matrices for one augmented matrix.

Try an example.

Example 11

Find the equivalent matrix to $\begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix}$ by successively performing the following operations.

- Multiply row 2 by $\frac{1}{2}$.

Solution:

$$\begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix} \begin{matrix} \textcircled{1} \\ \textcircled{2} \end{matrix}$$

$$\tfrac{1}{2}\times\textcircled{2}: \begin{bmatrix} 5 & -1 \\ 1 & 2 \end{bmatrix}$$

- Add -5 times row 2 to row 1.

  Solution:

  $$-5\times(2)+(1):\begin{bmatrix}0 & -11\\ 1 & 2\end{bmatrix}$$

- Interchange the rows.

  Solution:

  $$(1)\leftrightarrow(2):\begin{bmatrix}1 & 2\\ 0 & -11\end{bmatrix}$$

  Therefore, the matrix $\begin{bmatrix}5 & -1\\ 2 & 4\end{bmatrix}$ is row equivalent to $\begin{bmatrix}1 & 2\\ 0 & -11\end{bmatrix}$ according to the given instructions.

Example 12

Examine the following system of equations.

$$\left.\begin{aligned}x-2y-3z-5&=0\\ 2x-y+4z+7&=0\\ 5x+4y+z&=4\end{aligned}\right\}$$

- Write the matrix of coefficients.

  Solution:

  Note that the equations should be written so that the constant is on the right side of the equal sign.

  $$\begin{bmatrix}1 & -2 & -3\\ 2 & -1 & 4\\ 5 & 4 & 1\end{bmatrix}$$

- Write the augmented matrix.

  Solution:

  $$\begin{bmatrix}1 & -2 & -3 & 5\\ 2 & -1 & 4 & -7\\ 5 & 4 & 1 & 4\end{bmatrix}$$

Now try the practice exercise.

$(1)\leftrightarrow(2)$ means row 1 is interchanged with row 2.

Do any five of the following problems.

1. Examine the following system of equations.

$$\left.\begin{aligned} 4x-y&=5 \\ 2x+3y&=6 \end{aligned}\right\}$$

   a. Write the matrix of coefficients.

   b. Write the augmented matrix.

2. Write the system of equations for the augmented matrix if the variables are x, y, and z.

$$\begin{bmatrix} 1 & 4 & -1 & 2 \\ -2 & -1 & 4 & 6 \\ 3 & 5 & 3 & 5 \end{bmatrix}$$

3. Perform the following operations successively on the following matrix.

$$\begin{bmatrix} 2 & -1 & 2 \\ 1 & 5 & -3 \\ 3 & 4 & 1 \end{bmatrix}$$

   a. Interchange row 1 and row 3.

   b. Add -2 times row 2 to row 3.

   c. Add 2 times row 1 to row 2.

4. Complete the following chart by writing out the matrix of coefficients and the augmented matrix for each of the following systems of equations.

System of Equations	Matrix of Coefficients	Augmented Matrix
$x+y=8$ $2x-y=3$		
$4x-5y=-3$ $x+6y=10$		
$x+2y-z=2$ $2x-3y+z=5$ $3x+5y+6z=9$		

5. Perform the following operations on the given matrix.

$$\begin{bmatrix} 2 & -3 & 7 \\ 5 & 0 & -4 \end{bmatrix}$$

   a. Interchange the rows.

   b. Add 2 times row 1 to row 2.

   c. Multiply row 1 by 3.

6. Write the matrix of coefficients and the augmented matrix for the system of equations shown.

$$2x-y-3z=7$$
$$3x-4y+z=5$$
$$-x+3y+2z=10$$

7. Write the system of equations for the following matrices if the variables are x, y, and z.

a. $\begin{bmatrix} 4 & 2 & 3 & -4 \\ 2 & -3 & -1 & 0 \end{bmatrix}$

b. $\begin{bmatrix} 5 & 3 & 1 & 8 \\ 1 & 1 & 2 & -5 \\ 2 & -1 & 6 & 4 \end{bmatrix}$

8. Obtain the row-equivalent matrix by completing the given operations. (The first one is done as an example.)

$$\begin{bmatrix} 2 & -1 & 6 \\ 1 & 1 & 2 \\ 5 & 3 & 1 \end{bmatrix} \begin{matrix} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \end{matrix}$$

$\textcircled{1} \leftrightarrow \textcircled{3}$: $\begin{bmatrix} 5 & 3 & 1 \\ 1 & 1 & 2 \\ 2 & -1 & 6 \end{bmatrix}$

$2 \times \textcircled{2}$: $\begin{bmatrix} & & \end{bmatrix}$

$3 \times \textcircled{2} + \textcircled{3}$: $\begin{bmatrix} & & \end{bmatrix}$

$\textcircled{1} \leftrightarrow \textcircled{3}$: $\begin{bmatrix} & & \end{bmatrix}$

For solutions to **Extra Help**, turn to **Appendix A, Topic 3**.

Extensions

You know that a matrix is a display of numbers in rows and columns. This display is sometimes referred to as an array. When numbers are written in this form, they could be used to describe vectors in a rectangular space. Ever since matrices were introduced to mathematics in 1858, mathematicians have found various uses for them.

Matrices are widely used for inventory control in factories; quantum theory in physics; cost analysis in transportation and other industries; deployment problems in military operations; and data analysis in sociology and psychology.

As you know, matrices can be used to solve or verify systems of equations. As in every other aspect of mathematics, the study of matrices has rules and properties to which you have to adhere.

Now look at a few of the basic operations of matrices.

Equal Matrices

Two matrices are equal if they have the same dimensions and the same corresponding elements.

For example, $\begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ if only $a = 2$, $b = 3$, $c = 4$, and $d = 5$.

Therefore, the following matrices are **not** equal.

$$\begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix} \neq \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix} \neq \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$$

Example 13

Show that these matrices are equal.

- $\begin{bmatrix} 0 & 1 & 3 \\ 2 & -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 2\times 0 & \frac{2}{2} & \frac{6}{2} \\ \frac{6}{3} & -\frac{3}{3} & 0\times 4 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 0 & 1 & 3 \\ 2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 2\times 0 & \frac{2}{2} & \frac{6}{2} \\ \frac{6}{3} & -\frac{3}{3} & 0\times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 3 \\ 2 & -1 & 0 \end{bmatrix}$$

Therefore, the matrices are equal.

- $\begin{bmatrix} 9 & 3 \\ 2 & 4 \end{bmatrix}$ and $\begin{bmatrix} 3^2 & \frac{6}{2} \\ \sqrt{4} & 2^2 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 9 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 3^2 & \frac{6}{2} \\ \sqrt{4} & 2^2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 3 \\ 2 & 4 \end{bmatrix}$$

Therefore, the matrices are equal.

Matrix Transposition

A matrix is **transposed** when the rows and the columns are interchanged.

If you have matrix A, then the transposed form is A^T.

For example, if $A = \begin{bmatrix} 4 & 2 \\ 0 & 3 \end{bmatrix}$, then $A^T = \begin{bmatrix} 4 & 0 \\ 2 & 3 \end{bmatrix}$.

Look at the following example.

Example 14

Write the transposed matrix for each of the following.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \\ 4 & 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 4 & 5 \\ 1 & 2 & -3 \end{bmatrix}$$

Solution:

$$A^T = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 2 \\ -1 & 0 & 3 \end{bmatrix} \qquad B^T = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 4 & 2 \\ 3 & 5 & -3 \end{bmatrix}$$

Matrix Addition

Just as the sum of two real numbers is a unique real number, the sum of two matrices is also a unique matrix. This is called the **law of closure** for addition.

The way to add two matrices is to add the corresponding elements of the matrices.

Study the following example involving the addition of matrices.

Example 15

Find the sum of matrix A and matrix B where $A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 4 & 2 \\ 3 & 2 & 1 \end{bmatrix}$.

Solution:

$$\begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 4 & 2 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} (2+(-1)) & (3+4) & (1+2) \\ (-1+3) & (0+2) & (4+1) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 7 & 3 \\ 2 & 2 & 5 \end{bmatrix}$$

Only matrices having the same dimensions can be added. The sum of two matrices of different order or dimensions is yet undefined.

Multiplication by a Scalar

When you multiply a matrix by a scalar, the product is a new matrix. In order to do this you must multiply each element in the matrix by the scalar quantity.

Examine an example of this type.

Example 16

Find the scalar product of -3 and the given matrix.

$$\begin{bmatrix} 3 & 0 & -1 \\ 1 & -2 & 3 \\ -4 & 0 & 1 \end{bmatrix}$$

Solution:

$$(-3)\begin{bmatrix} 3 & 0 & -1 \\ 1 & -2 & 3 \\ -4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3(-3) & 0(-3) & (-1)(-3) \\ 1(-3) & (-2)(-3) & 3(-3) \\ (-4)(-3) & 0(-3) & 1(-3) \end{bmatrix}$$
$$= \begin{bmatrix} -9 & 0 & 3 \\ -3 & 6 & -9 \\ 12 & 0 & -3 \end{bmatrix}$$

Matrix Multiplication

It is quite interesting to see what you get when you multiply two matrices. This operation is best shown by an example.

Example 17

Multiply matrix A by matrix B where $A=\begin{bmatrix}3 & 4 & 2\end{bmatrix}$ and $B=\begin{bmatrix}2\\-3\\1\end{bmatrix}$.

Solution:

The product $A\times B$ means matrix A is on the left and matrix B is on the right.

$$A\times B=\begin{bmatrix}3 & 4 & 2\end{bmatrix}\begin{bmatrix}2\\-3\\1\end{bmatrix}=\begin{bmatrix}(3)(2)+(4)(-3)+(2)(1)\end{bmatrix}$$
$$=\begin{bmatrix}6-12+2\end{bmatrix}$$
$$=\begin{bmatrix}-4\end{bmatrix}$$

Example 18

Find the product of $A\times B$ where $A=\begin{bmatrix}2 & -1\\3 & 2\end{bmatrix}$ and $B=\begin{bmatrix}3 & 5\\-2 & 4\end{bmatrix}$.

Solution:

$$A\times B=\begin{bmatrix}2 & -1\\3 & 2\end{bmatrix}\begin{bmatrix}3 & 5\\-2 & 4\end{bmatrix}=\begin{bmatrix}(2)(3)+(-1)(-2) & (2)(5)+(-1)(4)\\(3)(3)+(2)(-2) & (3)(5)+(2)(4)\end{bmatrix}$$
$$=\begin{bmatrix}8 & 6\\5 & 23\end{bmatrix}$$

Now study this example.

Example 19

Determine the product $A\times B$ where $A=\begin{bmatrix}-4 & 2\\-8 & 4\end{bmatrix}$ and $B=\begin{bmatrix}4 & 6\\8 & 12\end{bmatrix}$.

Solution:

$$A\times B=\begin{bmatrix}-4 & 2\\-8 & 4\end{bmatrix}\begin{bmatrix}4 & 6\\8 & 12\end{bmatrix}$$
$$=\begin{bmatrix}(-4)(4)+(2)(8) & (-4)(6)+(2)(12)\\(-8)(4)+(4)(8) & (-8)(6)+(4)(12)\end{bmatrix}$$
$$=\begin{bmatrix}-16+16 & -24+24\\-32+32 & -48+48\end{bmatrix}$$
$$=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

This matrix $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ is referred to as the zero matrix and is denoted by 0.

This example shows that when you work with matrices, if $A\times B=0$, it does not mean that $A=0$ or $B=0$.

Now try the following exercise.

Do any six of these problems.

1. Determine the sum $A+B$ where $A=\begin{bmatrix}2 & 1 & 3\\ 1 & -1 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & 1 & 2\\ -1 & 1 & 2\end{bmatrix}$.

2. Why can't you add $\begin{bmatrix}0 & 1 & 2\\ 3 & -1 & 4\end{bmatrix}$ to $\begin{bmatrix}1 & 0 & 3\\ 3 & 0 & 2\\ 1 & 2 & 3\end{bmatrix}$?

3. If $A=\begin{bmatrix}3 & 2 & 1\\ 4 & 5 & 0\\ 2 & 3 & 6\end{bmatrix}$, then write the matrix for A^T.

4. If $A=\begin{bmatrix}1 & 3 & -1\end{bmatrix}$ and $B=\begin{bmatrix}2\\ 1\\ 3\end{bmatrix}$, find $A\times B$.

5. Transpose the following matrix.

$$\begin{bmatrix}1 & 2 & 0 & 3\\ 4 & -1 & -2 & 6\\ 0 & 3 & 1 & -4\\ -4 & 5 & 6 & -4\end{bmatrix}$$

6. Determine the product $A\times B$ where $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ and $B=\begin{bmatrix}u & v\\ w & x\end{bmatrix}$.

7. Calculate the product $A\times B$ where $A=\begin{bmatrix}1 & 2 & 3\\ 1 & 3 & 3\\ 1 & 2 & 4\end{bmatrix}$ and $B=\begin{bmatrix}6 & -2 & -3\\ -1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}$.

8. For the following matrices show that $A\times B=0$ where $A\neq 0$ and $B\neq 0$.

$$A=\begin{bmatrix}-2 & 1\\ -4 & 2\end{bmatrix} \qquad B=\begin{bmatrix}2 & 3\\ 4 & 6\end{bmatrix}$$

For solutions to **Extensions**, turn to **Appendix A, Topic 3**.

Unit Summary

What You Have Learned

In your study of systems of equations, you have learned

- there are three kinds of systems of linear equations, namely

  – independent systems having only one real and distinct solution
  – dependent systems having an infinite number of solutions
  – inconsistent systems having no solution

- there are three ways of solving systems of linear equations, namely

  – the comparison method
  – the elimination method
  – the substitution method

- in order to solve a system of linear equations, the number of unknowns should be equal to the number of equations

Unit Summary

- two important matrices can be formed from the coefficients of the variables and the constant terms of a system of equations
- a matrix can be transformed into a row-reduced echelon form by following certain simple procedures

You are now ready to

complete the **Unit Assignment**.

Appendices

Appendix A
Solutions

Review

Topic 1 Methods of Solving Systems of Linear Equations with Two Variables

Topic 2 Systems of Linear Equations with Three Variables

Topic 3 Matrices – Definitions and Properties

Appendix B
Graphing Materials

Graph Paper

x-*y*-*z* Axes

Appendix A
Solutions

Review

1. a. $5x+3(8-x)=36$

$5x+24-3x=36$

$5x-3x=36-24$

$2x=12$

$x=\frac{12}{2}$

$x=6$

b. $7x-4=2x+21$

$7x-2x=21+4$

$5x=25$

$x=5$

c. $2x-5-(x-3)=7$

$2x-5-x+3=7$

$2x-x=7+5-3$

$x=12-3$

$x=9$

2. a. $5(x-2)=25$

$x-2=5$

$x=5+2$

$x=7$

Verification:

LS	RS
$5(x-2)$	25
$5(7-2)$	25
25	25
LS =	RS

b. $x^2-x+4-(x-1)^2=0$

$x^2-x+4-\left(x^2-2x+1\right)=0$

$x^2-x+4-x^2+2x-1=0$

$-x+2x=1-4$

$x=-3$

Verification:

LS	RS
$x^2-x+4-(x-1)^2$	0
$(-3)^2-(-3)+4-(-3-1)^2$	0
$9+3+4-(-4)^2$	0
$16-16$	0
0	0
LS =	RS

c. $6x-4(5x+9)=4(2x-9)$

$6x-20x-36=8x-36$

$6x-20x-8x=36-36$

$-22x=0$

$x=0$

Verification:

LS	RS
$6x-4(5x+9)$	$4(2x-9)$
$6(0)-4[5(0)+9]$	$4[2(0)-9]$
-36	-36

LS = RS

3. a. $\frac{1}{3}x+\frac{1}{2}x=10$ (Multiply by 6 which is the LCD of 3 and 2.)

$\frac{6x}{3}+\frac{6x}{2}=6(10)$

$2x+3x=60$

$5x=60$

$x=12$

b. $\frac{3x}{4}=\frac{x}{3}+\frac{5}{6}$ (Multiply by the LCD of 12.)

$12\left(\frac{3x}{4}\right)=\frac{12x}{3}+\frac{(12)(5)}{6}$

$9x=4x+10$

$9x-4x=10$

$5x=10$

$x=2$

c. $\frac{-(x+5)}{2}+\frac{3x-3}{8}+4=0$ (Multiply by 8.)

$\frac{8(3)(x-1)}{8}-\frac{8(x+5)}{2}+8(4)=0$

$3x-3-4x-20+32=0$

$3x-4x=23-32$

$-1x=-9$

$x=9$

4. a. $x^2+3x+2=0$

$x^2+2x+x+2=0$

$x(x+2)+(x+2)=0$

$(x+1)(x+2)=0$

$x+1=0$ or $x+2=0$

$x=-1$ $x=-2$

b. $3x^2+2x=1$

$3x^2+2x-1=0$

$(3x-1)(x+1)=0$

$3x-1=0$ or $x+1=0$

$x=\frac{1}{3}$ $\quad$ $x=-1$

c. $x^3+21x^2+68x=0$

$x(x^2+21x+68)=0$

$x(x+4)(x+17)=0$

$x=0,\ x+4=0$ or $x+17=0$

$x=-4$ $\quad$ $x=-17$

5. a. $x+y=3$ ①

$x-y=1$ ②

① + ②: $2x=4$

$x=2$

Substitute $x=2$ in ①.

$2+y=3$

$y=3-2$

$y=1$

Verification for ①:

LS	RS
$x+y$	3
$2+1$	3
3	3

LS = RS

Verification for ②:

LS	RS
$x-y$	1
$2-1$	1
1	1

LS = RS

b. $y-3x=2y-x=20$

Write this as the following system:

$y-3x=20$ ①

$2y-x=20$ ②

$3\times$ ②: $6y-3x=60$ ③

$y-3x=20$ ①

③ − ①: $5y=40$

$y=8$

Substitute $y=8$ in ①.

$8-3x=20$

$-3x=20-8=12$

$x=-4$

Verification:

LS	RS
$y-3x$	20
$8-3(-4)$	20
20	20
LS =	RS

c. $x+y=11$ ①

$\frac{x}{2}+\frac{y}{3}=5$ ②

$6\times$②: $3x+2y=30$ ③

$-2\times$①: $-2x-2y=-22$ ④

③+④: $x=8$

Substitute $x=8$ in ①.

$8+y=11$

$y=3$

Verification for ①:

LS	RS
$x+y$	11
$8+3$	11
11	11
LS =	RS

Verification for ②:

LS	RS
$\frac{x}{2}+\frac{y}{3}$	5
$\frac{8}{2}+\frac{3}{3}$	5
$4+1$	5
5	5
LS =	RS

6. a. $x^2+y^2=130$ ①

$x^2-y^2=32$ ②

①+②: $2x^2=162$

$x^2=81$

$x=\pm 9$

Substitute $x^2=81$ in ①.

$81+y^2=130$

$y^2=49$

$y=\pm 7$

Therefore, $x=\pm 9$ and $y=\pm 7$.

b. $(x+y)^2 = 144$ (1)

$xy = 32$ (2)

Write (2) as $y = \frac{32}{x}$.

Substitute for y in (1).

$$\left(x + \frac{32}{x}\right)^2 = 144$$

Take the square roots of both sides.

$$x + \frac{32}{x} = \pm 12$$

$x^2 - 12x + 32 = 0$ or $x^2 + 12x + 32 = 0$

$(x-4)(x-8) = 0$ $\quad$ $(x+4)(x+8) = 0$

$x = 4$ or $x = 8$ $\quad$ $x = -4$ or $x = -8$

Substitute $x = 4$ in (2).

$4y = 32$

$y = 8$

Therefore, when $x = 4$, then $y = 8$.

Also substitute $x = 8$ in (2).

$8y = 32$

$y = 4$

Thus, when $x = 8$, then $y = 4$.

Similarly, when $x = -4$, $y = -8$, and when $x = -8$, then $y = -4$.

c. $a^2 - b^2 = 11$ (1)

$a + b = 11$ (2)

Write (1) as follows:

$(a+b)(a-b) = 11$ (3)

Substitute (2) in (3).

$11(a-b) = 11$

$a - b = 1$ (4)

$a + b = 11$ (2)

(4) + (2): $2a = 12$

$a = 6$

Substitute $a = 6$ in (2).

$6 + b = 11$

$b = 11 - 6$

$= 5$

Therefore, $a = 6$ and $b = 5$.

7. $x+y=10$ (1)
$y+z=8$ (2)
$x+z=12$ (3)

$$\begin{array}{rr} & x+y=10 \quad (1) \\ & z+y=8 \quad (2) \\ \hline (1)-(2): & x-z=2 \quad (4) \end{array}$$

$$\begin{array}{rr} & x+z=12 \quad (3) \\ & x-z=2 \quad (4) \\ \hline (3)+(4): & 2x=14 \\ & x=7 \end{array}$$

Substitute $x=7$ in (1).

$$\begin{aligned} 7+y&=10 \\ y&=10-7 \\ y&=3 \end{aligned}$$

Substitute $x=7$ in (3).

$$\begin{aligned} 7+z&=12 \\ z&=12-7 \\ z&=5 \end{aligned}$$

The solution to the system is $x=7$, $y=3$, and $z=5$.

Exploring Topic 1

Activity 1

Solve systems of linear equations with two unknowns.

1. a. $2x-y=7$ (1)
$5x+2y=13$ (2)

Write (1) and (2) as follows:

$y=2x-7$ (3)

$y=\frac{13}{2}-\frac{5x}{2}$ (4)

Equate equations (3) and (4).

$$\begin{aligned} 2x-7&=\frac{13}{2}-\frac{5x}{2} \\ 4x-14&=13-5x \\ 5x+4x&=13+14 \\ x&=3 \end{aligned}$$

Substitute in (3) to obtain the value for y.

$$y = 2x - 7$$
$$y = 2(3) - 7$$
$$= 6 - 7$$
$$y = -1$$

Therefore, $x = 3$ and $y = -1$.

b. $y = 2x + 3$ (1)
$x + y = 6$ (2)

Write (2) as $y = 6 - x$.

Equate equations (1) with (2).

$$2x + 3 = 6 - x$$
$$2x + x = 6 - 3$$
$$3x = 3$$
$$x = 1$$

Substitute $x = 1$ in (3).

$$y = 6 - 1$$
$$= 5$$

Therefore, $x = 1$ and $y = 5$.

2. a. $x + 2y = 15$ (1)
$3x - 2y = 9$ (2)

Write (1) as $x = 15 - 2y$. (3)

Substitute for x in (2).

$$3(15 - 2y) - 2y = 9$$
$$45 - 6y - 2y = 9$$
$$-8y = -45 + 9$$
$$y = \frac{9}{2}$$

Substitute $y = \frac{9}{2}$ in (1).

$$x + 2\left(\frac{9}{2}\right) = 15$$
$$x + 9 = 15$$
$$x = 15 - 9$$
$$x = 6$$

Therefore, $x = 6$ and $y = \frac{9}{2}$.

b. $2x - 5y = 6$ (1)
$5x + 2y = \frac{1}{2}$ (2)

Write (1) as $x = \frac{5}{2}y + 3$. (3)

Substitute ③ in ②.

$5\left(\frac{5}{2}y+3\right)+2y=\frac{1}{2}$ (Multiply through by 2.)

$25y+30+4y=1$

$29y=-29$

$y=-1$

Substitute $y=-1$ in ①.

$2x-5(-1)=6$

$2x=6-5$

$x=\frac{1}{2}$

Therefore, $x=\frac{1}{2}$ and $y=-1$.

3. a. $x-2y=-3$ ①

$2x+y=-2$ ②

$x-2y=-3$ ①

$2\times$②: $4x+2y=-4$ ③

①+③: $5x=-7$

$x=-\frac{7}{5}$

Substitute $x=-\frac{7}{5}$ in ② to find y.

$2\left(-\frac{7}{5}\right)+y=-2$

$y=\frac{14}{5}-2$

$y=\frac{4}{5}$

The solution is $x=-\frac{7}{5}$ and $y=\frac{4}{5}$.

b. $x+2y=15$ ①

$6x-4y=18$ ②

$x+2y=15$ ①

$\frac{1}{2}\times$②: $3x-2y=9$ ③

①+③: $4x=24$

$x=\frac{24}{4}$

$x=6$

Substitute $x=6$ in ①.

$6+2y=15$

$2y=15-6$

$y=\frac{9}{2}$

The solution is $x=6$ and $y=\frac{9}{2}$.

4. a. $3x = 2y$ (1)

$12y = -8x + 13$ (2)

Write (1) as $y = \frac{3}{2}x$.

Substitute for y in (1).

$$12\left(\frac{3x}{2}\right) = -8x + 13$$

$$18x = -8x + 13$$

$$26x = 13$$

$$x = \frac{1}{2}$$

Substitute $x = \frac{1}{2}$ in (2).

$$y = \frac{3}{2}\left(\frac{1}{2}\right)$$

$$y = \frac{3}{4}$$

Verification for (1):

LS		RS
$3x$		$2y$
$3\left(\frac{1}{2}\right)$		$2\left(\frac{3}{4}\right)$
$\frac{3}{2}$		$\frac{3}{2}$
LS	=	RS

Verification for (2):

LS		RS
$12y$		$-8x + 13$
$12\left(\frac{3}{4}\right)$		$-8\left(\frac{1}{2}\right) + 13$
9		9
LS	=	RS

Thus, the solution is $x = \frac{1}{2}$ and $y = \frac{3}{4}$.

b. $x + \frac{1}{3}y = y$ (1)

$y + \frac{1}{4}x = x + 3$ (2)

Simplify (1) and (2) to read as follows:

$y = \frac{3}{2}x$ (3)

$y = \frac{3}{4}x + 3$ (4)

Equate (3) and (4).

$$\frac{3}{2}x = \frac{3}{4}x + 3$$

$$\frac{3}{2}x - \frac{3}{4}x = 3$$

$$\left(\frac{3}{2} - \frac{3}{4}\right)x = 3$$

$$\frac{3}{4}x = 3$$

$$x = 4$$

Substitute $x = 4$ in (4).

$$y = \frac{3}{4}x + 3$$
$$= \frac{3}{4}(4) + 3$$
$$= 3 + 3$$
$$= 6$$

Verification for (1):

LS	RS
$x + \frac{1}{3}y$	6
$4 + \frac{1}{3}(6)$	6
$4 + 2$	6
6	6
LS =	RS

Verification for (2):

LS	RS
$y + \frac{1}{4}x$	$x + 3$
$6 + \frac{1}{4}(4)$	$4 + 3$
7	7
LS =	RS

Thus, the solution is $x = 4$ and $y = 6$.

Activity 2

Classify systems of linear equations in such categories as independent, dependent, or inconsistent.

1. $3x + 7y = 24$ (1)
   $5x + 2y = 11$ (2)

$2 \times (1)$: $6x + 14y = 48$ (3)
$7 \times (2)$: $35x + 14y = 77$ (4)
(3) − (4): $-29x = -29$
$x = 1$

(Note that $x = 1$ and $3x + 7y = 24$ are an equivalent reduced system.)

Substitute $x = 1$ in (1).

$$3(1) + 7y = 24$$
$$y = 3$$

The solution is $x = 1$ and $y = 3$.

2. a. $2x - 5y = 6$ (1)
   $5x + 2y = \frac{1}{2}$ (2)

$5 \times (1)$: $10x - 25y = 30$ (3)
$-2 \times (2)$: $-10x - 4y = -1$ (4)
(3) + (4): $-29y = 29$
$y = -1$

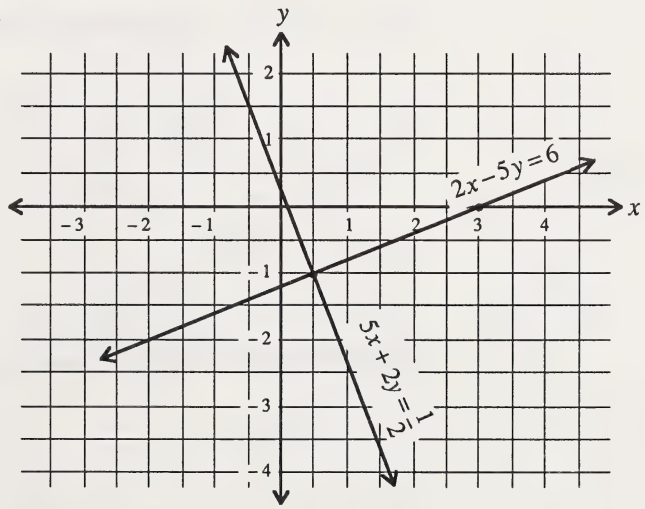
Therefore, $y=-1$ and $2x-5y=6$ are an equivalent reduced system.

Substitute $y=-1$ in (1).

$$2x-5(-1)=6$$
$$2x=6-5$$
$$x=\frac{1}{2}$$

The solution is $x=\frac{1}{2}$ and $y=-1$.

b.



c. Since the two lines intersect, the system is independent.

3. a. $x-2y=-3$ (1)
$2x+y=-2$ (2)

$2\times$(2): $4x+2y=-4$ (3)
$x-2y=-3$ (1)
(3)+(1): $5x=-7$
$x=-\frac{7}{5}$

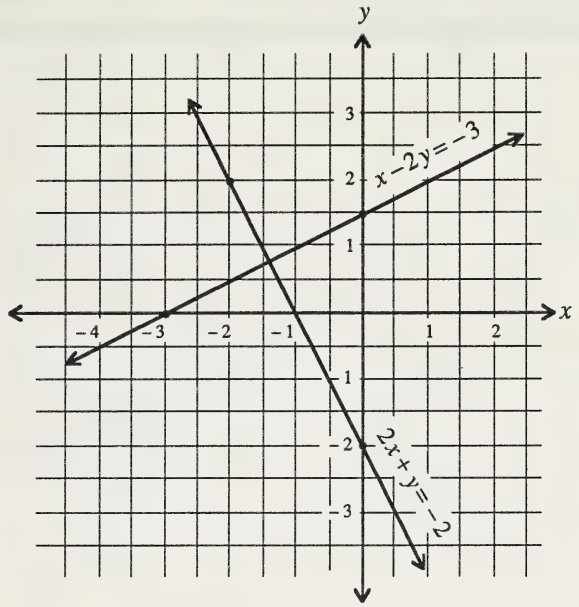
$x=-\frac{7}{5}$ and $x-2y=-\frac{5}{3}$ are an equivalent reduced system.

Substitute $x=-\frac{7}{5}$ in (2).

$$y=-2\left(-\frac{7}{5}\right)-2$$
$$=\frac{14}{5}-\frac{10}{5}$$
$$=\frac{4}{5}$$

The solution is $x=-\frac{7}{5}$ and $y=\frac{4}{5}$.

b.



c. Since the lines intersect, the system is independent.

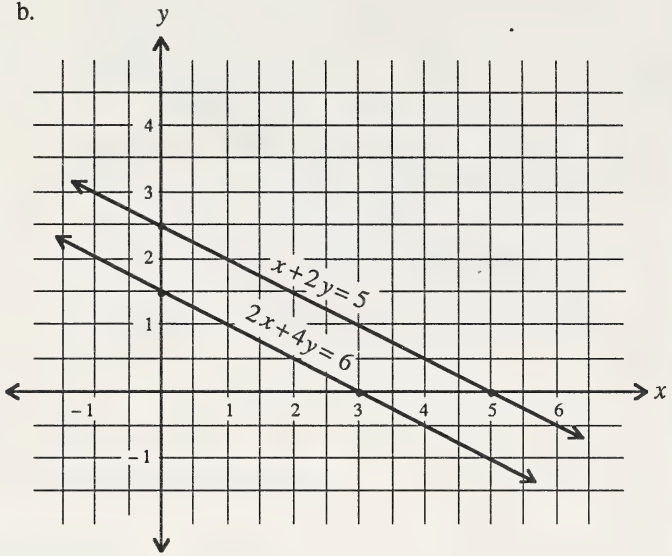
4. a. $x+2y=5$ (1)
$2x+4y=6$ (2)

$-2\times$(1): $-2x-4y=-10$ (3)
$2x+4y=6$ (2)
(3)+(2): $0x+0y=-4$ (4)

Equations (2) and (4) are an equivalent reduced system.

Since $0\neq -4$ as in (4), the solution is ϕ. There are no values that will substitute for x and y to make the equations true.

b.



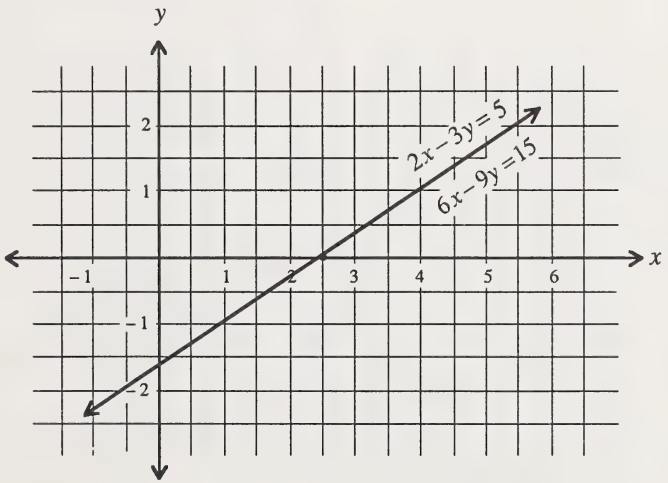
c. Since the two lines are parallel, the system is inconsistent.

5. a. $2x-3y=5$ (1)
$6x-9y=15$ (2)

$3\times$(1): $6x-9y=15$ (3)
$6x-9y=15$ (2)
(3)+(2): $0x-0y=0$ (4)

Equations (1) and (4) are an equivalent reduced system.

Any value of x and y will satisfy (4). Therefore, this system has an infinite number of solutions.



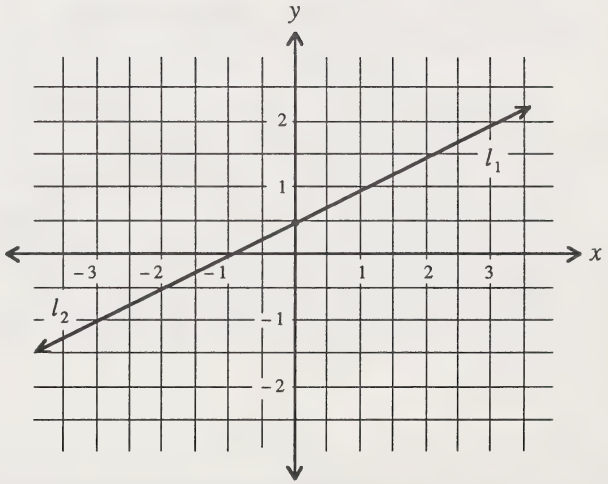
c. Since any value of x and y will satisfy (4), the system has an infinite number of solutions.

If you let $y=k$, then $x=\frac{5+3k}{2}$. The solution set is $\left(\frac{5+3k}{2}, k\right)$ where k is any real number.

The graph is two concurrent straight lines; therefore, the system is dependent.

Extra Help

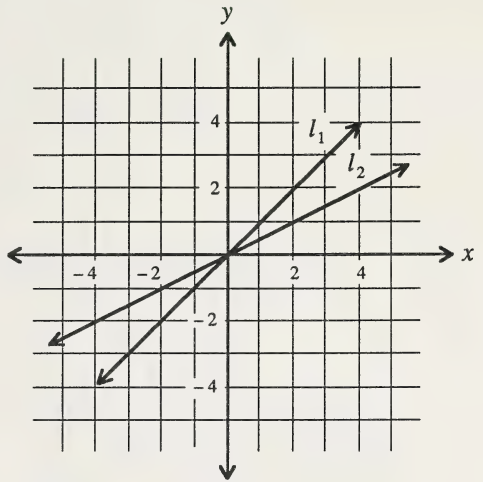
1. a. l_1: $2y=x+1$
l_2: $2x+2=4y$



This system is dependent.

b. l_1: $x = y$

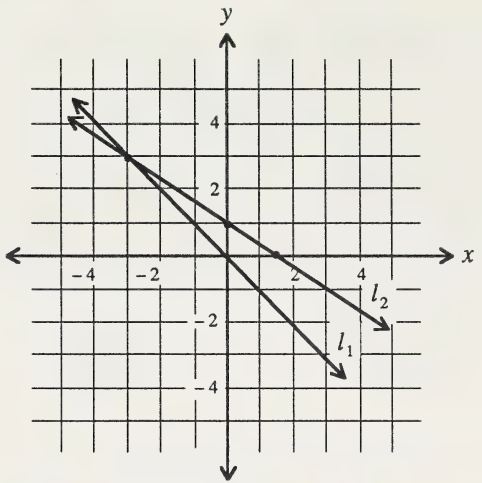
l_2: $x = 2y$



This system is independent.

c. l_1: $x + y = 0$

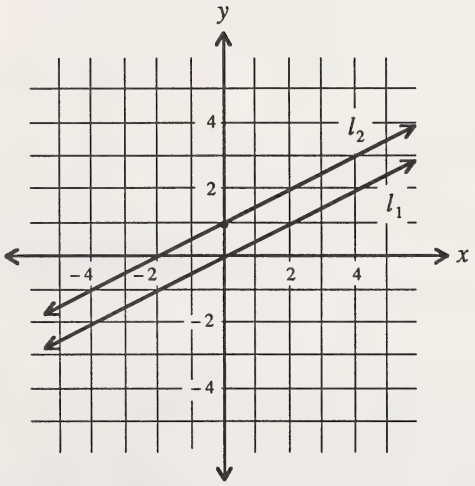
l_2: $2x + 3y = 3$



This system is independent.

d. $l_1: x-2y=0$

$l_2: 2y-x=2$



This system is inconsistent.

2. a. $l_1: x+2y=3$ (1)

$l_2: 3y+4x=2$ (2)

$-4\times(1): -4x-8y=-12$ (3)

$4x+3y=2$ (2)

(3)+(2): $-5y=-10$

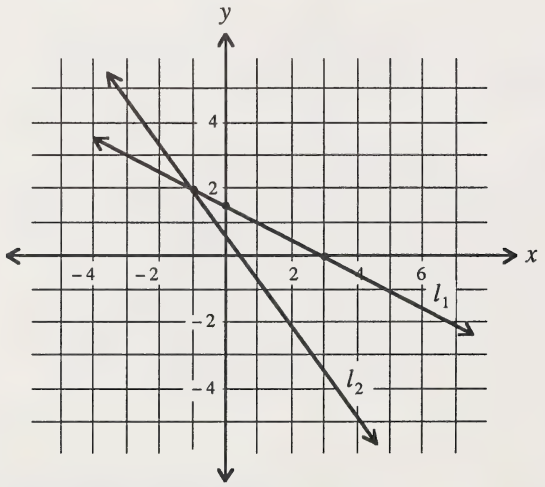
$y=2$

Substitute $y=2$ in (1).

$x+2(2)=3$

$x=-1$

Therefore, the solution is $x=-1$ and $y=2$.



This system is independent.

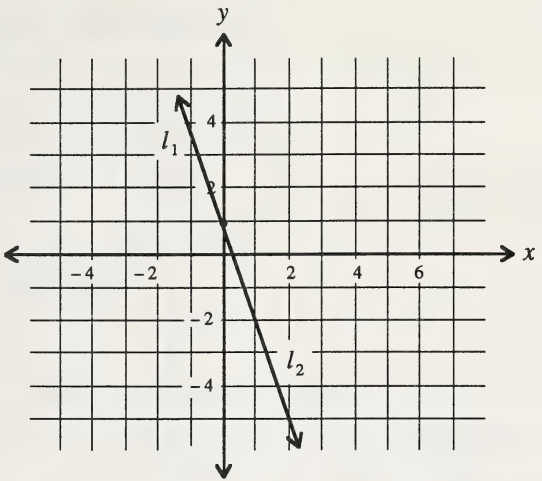
b. l_1: $3x+y=1$ (1)

l_2: $2y+6x=2$ (2)

$2\times$(1): $6x+2y=2$ (3)

$6x+2y=2$ (2)

(3) − (2): $0x+0y=0$



The system of equations is dependent.

Extensions

1. Let x represent Heidi's share of the lottery and y represent Dimetri's share.

$x+y=1000$ (1)

$x-2y=100$ (2)

(1) − (2): $3y=900$

$y=300$

Substitute $y=300$ in (1).

$x+300=1000$

$x=1000-300$

$x=700$

Heidi received \$700 and Dimetri received \$300.

2. Let the boy's age be x and his father's age be y.

$y-x=20$ (1)

$2y-40=2x$ (2)

$\frac{1}{2}\times$(2): $y-20=x$

$y-x=20$ (4)

$$y - x = 20 \quad (1)$$
$$y - x = 20 \quad (4)$$
$$(1) - (4): \quad 0y + 0x = 0 \quad (5)$$

You can not solve for x or y since (4) is the same as (1). The statement is redundant. However, (5) shows that the two statements are consistent.

3. Let the length of the plot be x metres, and let y represent the width in metres. From the perimeter formula and the Pythagorean theorem, you can obtain the following equations:

$$2x + 2y = 98 \quad (1)$$
$$x^2 + y^2 = 1681 \quad (2)$$
$$\tfrac{1}{2} \times (1): \quad x + y = 49$$
$$y = 49 - x$$

Substitute for y in (2).

$$x^2 + (49 - x)^2 = 1681$$
$$x^2 + 49^2 - 98x + x^2 = 1681$$
$$2x^2 - 98x + 720 = 0$$
$$x^2 - 49x + 360 = 0$$
$$(x - 9)(x - 40) = 0$$
$$x = 9 \text{ or } x = 40$$

Since the length is greater than the width, you ignore $x = 9$.

Substitute $x = 40$ in (2).

$$y = 49 - 40$$
$$= 9$$

The length of the lot is 40 m and the width is 9 m.

4. Let x be the amount deposited in the first account and y be the amount deposited in the second account.

$$0.13x + 0.04y = 120 \quad (1)$$
$$x + y = 1200 \quad (2)$$

$$100 \times (1): \quad 13x + 4y = 12\,000 \quad (3)$$
$$-4 \times (2): \quad -4x - 4y = -4800 \quad (4)$$
$$(3) + (4): \quad 9x = 7200$$
$$x = \frac{7200}{9}$$
$$= 800$$

Substitute $x = 800$ in (2).

$$800 + y = 1200$$
$$y = 1200 - 800$$
$$= 400$$

The amount deposited in the first account was \$800, and the amount deposited in the second account was \$400.

5. Let the amount of 20% ammonia be x. Let the amount of 10% ammonia be y.

Equating the strengths of ammonia, you can obtain the following equations:

$x+y=100$ (1)

$0.2x+0.1y=(0.17)(100)$ (2)

Manipulate (2) to obtain (3).

$2x+y=170$ (3)

$x+y=100$ (1)

(3) − (1): $x=70$

Substitute $x=70$ in (1).

$70+y=100$

$y=30$

There were 70 L of 20% ammonia and 30 L of 10% ammonia.

Exploring Topic 2

Activity 1

Define systems of linear equations as homogeneous or nonhomogeneous and solve the systems.

1. $x+y=10$ (1)

$y+z=8$ (2)

$x+z=12$ (3)

(2) − (1): $z-x=-2$ (4)

$z+x=12$ (3)

(4) + (3): $2z=10$

$z=5$

Substitute $z=5$ in (2).

$y+5=8$

$y=3$

Substitute $y=3$ in (1).

$x+3=10$

$x=7$

The solution is $x=7$, $y=3$, and $z=5$.

2. $5+4z=y+3$ (1)
$x+y=3z$ (2)
$7z=x+2$ (3)

(1) − (2): $x+z=-2$ (4)

$7z-x=2$ (3)

(4) + (3): $8z=0$

$z=0$

Substitute $z=0$ in (1).

$5+4z=y+3$

$5+4(0)=y+3$

$y=2$

Substitute $z=0$ in (3).

$7z=x+2$

$7(0)=x+2$

$x=-2$

The solution is $x=-2$, $y=2$, and $z=0$.

Verification for (1):

LS	RS
$5+4z$	$y+3$
$5+4(0)$	$2+3$
5	5
LS =	RS

Verification for (2):

LS	RS
$x+y$	$3z$
$-2+2$	$3(0)$
0	0
LS =	RS

Verification for (3):

LS	RS
$7z$	$x+2$
$7(0)$	$-2+2$
0	0
LS =	RS

3. If the system of equations is independent, then there is a distinct and unique solution to the system.

$x+y=14$ (1)
$y+z=9$ (2)
$z+x=11$ (3)

$x+y=14$ (1)

$y+z=9$ (2)

(1) − (2): $x-z=5$ (4)

$$\begin{array}{rr} z+x=11 & (3) \\ -z+x=5 & (4) \\ \hline 2x=16 & \end{array}$$

(3) + (4):

$x=8$

Substitute $x=8$ in (1).

$8+y=14$

$y=6$

Substitute $x=8$ in (2).

$z+8=11$

$z=3$

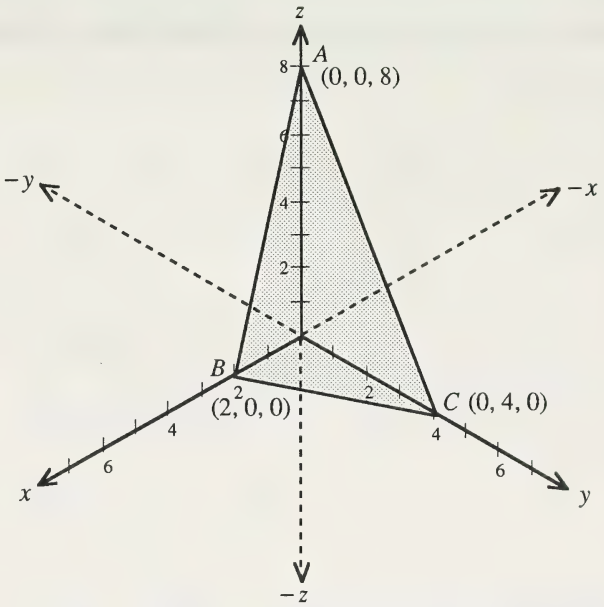
The distinct and unique solution is $x=8$, $y=6$, and $z=3$; therefore, the system is independent.

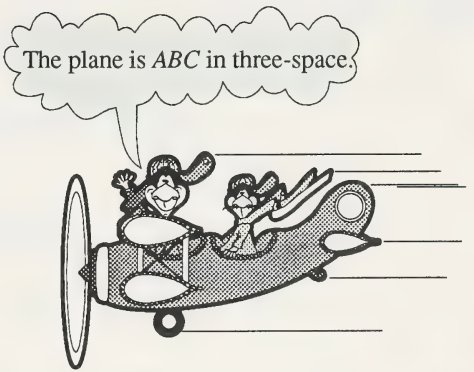
4. Since there are three variables in the equation $4x+2y+z=8$, then the plane exists in three-space.

If you let $x=y=0$, then $z=8$; thus, plot $(0,0,8)$.

If you let $x=z=0$, then $y=4$; thus, plot $(0,4,0)$.

If you let $y=z=0$, then $x=2$; thus, plot $(2,0,0)$.





5. $x+y+z=1$ (1)
$2x+3y+z=0$ (2)
$x-y+2z=4$ (3)

$-2\times(1)$: $-2x-2y-2z=-2$ (4)
$2x+3y+z=0$ (2)
(4)+(2): $y-z=-2$ (5)

(1)−(3): $2y-z=-3$ (6)
(6)−(5): $y=-1$

Substitute $y=-1$ in (4).

$-1-z=-2$
$z=1$

Substitute $y=-1$ and $z=1$ in (1).

$x-1+1=1$
$x=1$

Since the solution is $x=1$, $y=-1$, and $z=1$, the system of equations is independent.

6. a. $x+2y=0$ (1)
$4x-y=0$ (2)

$2\times(2)$: $8x-2y=0$ (3)
$x+2y=0$ (1)
(3)+(1): $9x=0$
$\therefore x=0$

Substitute $y=0$ in (1).

$y=0$

The solution is $x=0$ and $y=0$ or $(0,0)$. This point is unique and distinct; therefore, the system is independent.

b. $x-y+2z=0$ (1)
$3x-2y-5z=0$ (2)

$-3\times(1)$: $-3x+3y-6z=0$ (3)
$3x-2y-5z=0$ (2)
(3)+(2): $y-11z=0$ (4)

Note that equations ① and ④ are an equivalent reduced system.

Let $z = k$ where k is a real number.

Substitute $z = k$ and $y = 11k$ in ①.

$$x - 11k + 2k = 0$$
$$x = 9k$$

The solution to the system in terms of k is $x = 9k$, $y = 11k$, and $z = k$ where k is a real number; therefore, the system is dependent.

c. $x + 2y - z = 0$ ①
$2x + 4y - 2z = 0$ ②
$-3x - 6y + 3z = 0$ ③

$2 \times ① - ②$: $0x + 0y + 0z = 0$ ④
$3 \times ① + ③$: $0x + 0y + 0z = 0$ ⑤

$$\left.\begin{aligned} x + 2y - z &= 0 \\ 0y + 0z &= 0 \\ 0z &= 0 \end{aligned}\right\}$$ Thus, is an equivalent reduced system.

Let $z = l$ and $y = k$ where l and k are real numbers.

Substitute for y and z in ①.

$$x + 2k - l = 0$$
$$x = l - 2k$$

The solution is $x = l - 2k$, $y = k$, and $z = l$; therefore, the system is dependent.

Extra Help

1. $5x + 4y = 8$ ①
$2x - y = 11$ ②

$4 \times ②$: $8x - 4y = 44$ ③
$5x + 4y = 8$ ①
③ + ①: $13x = 52$
$x = 4$ ④

Substitute $x=4$ in (1).

$5(4)+4y=8$

$4y=8-20$

$y=-3$

Equations (1) and (4) are the equivalent reduced system.

The solution is $(4,-3)$; therefore, the system is independent.

2. $8x-4y=5$ (1)
   $4x-2y=8$ (2)

$-2\times$(2): $-8x+4y=-16$ (3)

$8x-4y=5$ (1)

(3)+(1): $0x+0y=-11$ (4)

No value of x or y will satisfy (4). Therefore, the solution is ϕ and the system is inconsistent.

3. $x-y=-3$ (1)
   $x+2y=12$ (2)

$x+2y=12$ (2)

$x-y=-3$ (1)

(2)−(1): $3y=15$

$y=5$ (3)

Substitute $y=5$ in (1).

$x-5=-3$

$x=2$

Equations (1) and (3) are an equivalent reduced system.

The solution is $x=2$ and $y=5$ or $(2,5)$; therefore, the system is independent.

4. $x-2y+4z=3$ (1)
$2x+y-2z=5$ (2)
$4x-3y+6z=7$ (3)

$-2\times(1)$: $-2x+4y-8z=-6$ (4)
$2x+y-2z=5$ (2)
(4)+(2): $0x+5y-10z=-1$ (5)

$-4\times(1)$: $-4x+8y-16z=-12$ (6)
$4x-3y+6z=7$ (3)
(6)+(3): $0x+5y-10z=-5$ (7)

(7)−(5): $0x+0y+0z=-4$

Equations (5) and (7) show signs of inconsistency. No values of x, y, and z will satisfy the system. Therefore, the solution is ϕ and the system is inconsistent.

5. $x+2y-z=3$ (1)
$6x-2y+5z=6$ (2)
$2x+y+z=4$ (3)

$-6\times(1)+(2)$: $0x-14y+11z=-12$ (4)
$-2\times(1)+(3)$: $0x-3y+32=-2$ (5)

$\frac{-14}{3}\times(5)+(4)$: $0x+0y+-3z=\frac{-8}{3}$ (6)

$-\frac{1}{14}\times(4)$: $0x+y-\frac{11}{14}z=\frac{12}{14}$

$-\frac{1}{3}\times(6)$: $0x+0y+z=\frac{8}{9}$

$z=\frac{8}{9}$

The equivalent reduced system is as follows:

$$\left.\begin{aligned} x+2y-z&=3\\ 0x+y-\frac{11}{14}z&=\frac{12}{14}\\ 0x+0y+z&=\frac{8}{9}\end{aligned}\right\}$$

The solution is $x=\frac{7}{9}$, $y=\frac{14}{9}$, and $z=\frac{8}{9}$ or $\left(\frac{7}{9},\ \frac{14}{9},\ \frac{8}{9}\right)$. This is an independent system.

6. $x+2y-z=3$ (1)
$6x-y+z=4$ (2)
$4x-5y+3z=-2$ (3)

It is not possible to look at this system at a glance and tell if it is dependent, independent, or inconsistent. In order to find out, this is what should be done.

$-6\times$(1): $-6x-12y+6z=-18$ (4)
$6x-y+z=4$ (2)
(4)+(2): $0x-13y+7z=-14$ (5)

$-4\times$(1)+(3): $0x-13y+7z=-14$ (6)

Note that (5)=(6).

$0x-13y+7z=-14$ (5)
$0x-13y+7z=-14$ (6)
(5)−(6): $0x+0y+0z=0$ (7)

The equivalent reduced system is as follows:

$x+2y-z=3$
$0x-13y+7z=-14$
$0x+0y+0z=0$

Any values of x, y, and z will satisfy (7). Therefore, let $z=k$, where k is any real number, and solve (6) for y.

$-13y+7k=14$

$$y=\frac{7k+14}{13}$$

The solutions is $x=\frac{11-k}{13}$, $y=\frac{7k+14}{13}$, and $z=k$.

Since there are an infinite number of solutions, the system is dependent.

Extensions

1. Denote the fraction by $\frac{x}{y}$. Then, from the given information you can obtain the following equations.

$\frac{x+4}{y+4}=\frac{7}{9}$ (1)
$\frac{x-3}{y-3}=\frac{3}{4}$ (2)

Manipulate (1) to read as follows:

$9(x+4)=7(y+4)$
$9x+36=7y+28$
$9x-7y=-8$ (3)

Write (2) as follows:

$4(x-3)=3(y-3)$

$4x-12=3y-9$

$4x-3y=3$ (4)

Substitute $y=59$ in (3).

$x=45$

Therefore, the fraction is $\frac{45}{59}$.
(Note: You may wish to verify the answer.)

2. a. Draw triangle ABC.

Let $AB=x$, $BC=y$, and $AC=z$. Then, from the given information you can obtain the following equations.

$x+y+z=30$ (1)

$x+y=17$ (2)

$y+z=25$ (3)

$z+x=18$ (4)

Since you have four equations and only three variables, you can arrive at the solutions much faster.

Substitute $x+y=17$ in (1) to find z.

$z=30-17$

$=13$

Substitute $y+z=25$ in (1).

$x=5$

Substitute $z+x=18$ in (1).

$y=12$

Therefore, $AB=5$ m, $BC=12$ m, and $AC=13$ m.

b. The ratio 5:12:13 for the sides of a triangle implies a right triangle, and the area is given as follows:

$A=\frac{1}{2}bh$

$=\frac{1}{2}(5)(12)$

$=30$

Therefore, the are is 30 m^2.

3. Let the three numbers be x, y, and z.

$x+y+z=72$ (1)

$\frac{x+z}{2}=y$ (2)

$\frac{y+2x}{2}=z$ (3)

Write equations (2) and (3) as follows:

$x+z=2y$ (4)

$y+2x=2z$ (5)

$2\times(4)$: $2x+2z-4y=0$

$2x-2z+y=0$ (5)

$2\times(4)+(5)$: $4x-3y=0$ (6)

$x+y+z=72$ (1)

$x-2y+z=0$ (4)

(1) − (4): $3y=72$

$y=24$

Substitute $y=24$ in (6).

$4x-3(24)=0$

$4x=72$

$x=18$

Substitute $x=18$ and $y=24$ in (1) to find z.

$z=72-18-24$

$=72-42$

$=30$

Therefore, the three numbers are 18, 24, and 30.
(You may wish to verify your answer.)

4. Let the three numbers be x, y, and z.

$x+y=30$ (1)

$y+z=36$ (2)

$z+x=32$ (3)

$y+z=36$ (2)

$y+x=30$ (1)

(2) − (1): $z-x=6$ (4)

$z+x=32$ (3)

$z-x=6$ (4)

(3) + (4): $2z=38$

$z=19$

Substitute $z=19$ in (2) to find y.

$y+19=36$

$y=36-19$

$y=17$

Substitute $y=17$ in (1) to find x.

$x+17=30$

$x=30-17$

$x=13$

Therefore, the three numbers are 13, 17, and 19.

5. Let there be x parts of gold, y parts of silver, and z parts of copper.

$x+y+z=18$ (1)
$75x+65y+50z=1110$ (2)
$20x+15y+24z=372$ (3)

$15\times(1)$: $15x+15y+15z=270$ (4)
$\frac{1}{5}\times(2)$: $15x+13y+10z=222$ (5)
$(4)-(5)$: $2y+5z=48$ (6)

$20x+15y+24z=372$ (3)
$-20\times(1)$: $-20x-20y-20z=-360$ (7)
$(3)+(7)$: $-5y+4z=12$ (8)

$5\times(6)+2\times(8)$: $33z=264$
$z=8$

Substitute $z=8$ in (6).

$2y+5(8)=48$
$2y=48-40$
$y=4$

Substitute $y=4$ and $z=8$ in (1).

$x+4+8=18$
$x=6$

Therefore, there are 6 parts of gold, 4 parts of silver, and 8 parts of copper.

Exploring Topic 3

Activity 1

Define matrices and their related properties.

1. a. matrix of coefficients $=\begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix}$

augmented matrix $=\begin{bmatrix}1 & 1 & 8\\ 2 & 1 & 7\end{bmatrix}$

b. matrix of coefficients $=\begin{bmatrix}4 & -5\\ 2 & 7\end{bmatrix}$

augmented matrix $=\begin{bmatrix}4 & -5 & -3\\ 2 & 7 & 14\end{bmatrix}$

c. matrix of coefficients $= \begin{bmatrix} 1 & 2 & 1 \\ 2 & -3 & -1 \\ 3 & 5 & 4 \end{bmatrix}$

augmented matrix $= \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & -3 & -1 & 6 \\ 3 & 5 & 4 & 5 \end{bmatrix}$

2. a. $\begin{bmatrix} 3 & -1 & 7 \\ 5 & 2 & 4 \end{bmatrix}$ ① ②

$① \leftrightarrow ②: \begin{bmatrix} 5 & 2 & 4 \\ 3 & -1 & 7 \end{bmatrix}$

b. $4 \times ① + ②: \begin{bmatrix} 5 & 2 & 4 \\ 23 & 7 & 23 \end{bmatrix}$

c. $6 \times ①: \begin{bmatrix} 30 & 12 & 24 \\ 23 & 7 & 23 \end{bmatrix}$

3. a. $\begin{bmatrix} 2 & -7 \\ 5 & 0 \end{bmatrix}$ ① ②

$3 \times ② + ①: \begin{bmatrix} 17 & -7 \\ 5 & 0 \end{bmatrix}$

b. $① \leftrightarrow ②: \begin{bmatrix} 5 & 0 \\ 17 & -7 \end{bmatrix}$

c. $3 \times ②: \begin{bmatrix} 5 & 0 \\ 51 & -21 \end{bmatrix}$

4. a. $\begin{bmatrix} 3 & -1 & 2 \\ 5 & 1 & 2 \\ 0 & 1 & 5 \end{bmatrix}$ ① ② ③

$① \leftrightarrow ③: \begin{bmatrix} 0 & 1 & 5 \\ 5 & 1 & 2 \\ 3 & -1 & 2 \end{bmatrix}$

b. $2 \times ②: \begin{bmatrix} 0 & 1 & 5 \\ 10 & 2 & 4 \\ 3 & -1 & 2 \end{bmatrix}$

c. $3 \times ② + ③: \begin{bmatrix} 0 & 1 & 5 \\ 10 & 2 & 4 \\ 33 & 5 & 14 \end{bmatrix}$

d. $① \leftrightarrow ②: \begin{bmatrix} 10 & 2 & 4 \\ 0 & 1 & 5 \\ 33 & 5 & 14 \end{bmatrix}$

5. $\begin{bmatrix} 3 & -2 & 7 \\ 1 & 4 & -2 \\ 2 & -1 & 3 \end{bmatrix}$ ① ② ③

$-1\times(3)+(1)$: $\begin{bmatrix}1 & -1 & 4\\ 1 & 4 & -2\\ 2 & -1 & 3\end{bmatrix}$

$(1)\leftrightarrow(2)$: $\begin{bmatrix}1 & -1 & 4\\ 2 & -1 & 3\\ 1 & 4 & -2\end{bmatrix}$

$4\times(2)$: $\begin{bmatrix}1 & -1 & 4\\ 8 & -4 & 12\\ 1 & 4 & -2\end{bmatrix}$

$\frac{1}{2}\times(1)+(2)$: $\begin{bmatrix}1 & -1 & 4\\ \frac{17}{2} & \frac{-9}{2} & 14\\ 1 & 4 & -2\end{bmatrix}$

6. $\begin{bmatrix}1 & 3 & -1\\ 2 & 4 & 6\end{bmatrix}$ $\begin{matrix}(1)\\(2)\end{matrix}$

$-2\times(1)+(2)$: $\begin{bmatrix}1 & 3 & -1\\ 0 & -2 & 8\end{bmatrix}$

$\frac{-1}{2}\times(2)$: $\begin{bmatrix}1 & 3 & -1\\ 0 & 1 & -4\end{bmatrix}$

7. $\begin{bmatrix}3 & -1 & 2\\ -1 & 4 & 9\end{bmatrix}$ $\begin{matrix}(1)\\(2)\end{matrix}$

$(1)\leftrightarrow(2)$: $\begin{bmatrix}-1 & 4 & 9\\ 3 & -1 & 2\end{bmatrix}$

$-1\times(1)$: $\begin{bmatrix}1 & -4 & -9\\ 3 & -1 & 2\end{bmatrix}$

$-3\times(1)+(2)$: $\begin{bmatrix}1 & -4 & -9\\ 0 & 11 & 29\end{bmatrix}$

$\frac{1}{11}\times(2)$: $\begin{bmatrix}1 & -4 & -9\\ 0 & 1 & \frac{29}{11}\end{bmatrix}$

8. $\begin{bmatrix}2 & 4 & 6\\ 5 & 2 & 3\end{bmatrix}$ $\begin{matrix}(1)\\(2)\end{matrix}$

$\frac{1}{2}\times(1)$: $\begin{bmatrix}1 & 2 & 3\\ 5 & 2 & 3\end{bmatrix}$

$-5\times(1)+(2)$: $\begin{bmatrix}1 & 2 & 3\\ 0 & -8 & -12\end{bmatrix}$

$\frac{-1}{8}\times(2)$: $\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & \frac{12}{8}\end{bmatrix}$

9. First change the system to its corresponding augmented matrix.

$$\begin{bmatrix} 1 & 4 & -3 \\ 3 & -2 & 1 \\ 2 & -1 & 3 \end{bmatrix} \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

$$-3\times(1)+(2): \begin{bmatrix} 1 & 4 & -3 \\ 0 & -14 & 10 \\ 2 & -1 & 3 \end{bmatrix}$$

$$\tfrac{-1}{14}\times(2): \begin{bmatrix} 1 & 4 & -3 \\ 0 & 1 & \frac{-5}{7} \\ 2 & -1 & 3 \end{bmatrix}$$

$$-2\times(1)+(3): \begin{bmatrix} 1 & 4 & -3 \\ 0 & 1 & \frac{-5}{7} \\ 0 & -9 & 9 \end{bmatrix}$$

$$9\times(2)+(3): \begin{bmatrix} 1 & 4 & -3 \\ 0 & 1 & \frac{-5}{7} \\ 0 & 0 & \frac{18}{7} \end{bmatrix}$$

$$\tfrac{7}{18}\times(3): \begin{bmatrix} 1 & 4 & -3 \\ 0 & 1 & \frac{-5}{7} \\ 0 & 0 & 1 \end{bmatrix}$$

Activity 2

Solve systems of equations using matrices.

1. a. $\left.\begin{aligned} x+3y&=-2 \\ y&=5 \end{aligned}\right\}$

$\therefore x=-17$

The solution set is $\{(-17,5)\}$; therefore, the system is independent.

b. $\left.\begin{aligned} x+5y&=4 \\ 0x+0y&=0 \end{aligned}\right\}$

Let $y=k$.

$\therefore x=-5k+4$

The solution set is $\{(-5k+4,k)\,|\,k\in R\}$; therefore, the system is dependent.

c. $\left.\begin{aligned} x+2y-3z&=1 \\ y&=2 \\ z&=4 \end{aligned}\right\}$

$\therefore x=1-(2)(2)+(3)(4)$

$=9$

The solution set is $\{(9,2,4)\}$; therefore, the system is independent.

d. $\left.\begin{aligned} x+5y+7z&=-7\\ 4z&=3\\ 0z&=1 \end{aligned}\right\}$

The solution set is ϕ; therefore, the system is inconsistent.

e. $\left.\begin{aligned} x+2y+5z&=3\\ z&=4\\ 0x+0y+0z&=0 \end{aligned}\right\}$

Let $y=k$.

$\therefore x=-2k-17$

The solution set is $\{(-2k-17,k,4)\,|\,k\in R\}$; therefore, the system dependent.

f. $\left.\begin{aligned} x+2y+5z&=3\\ y+z&=4 \end{aligned}\right\}$

Let $z=k$, and $y=4-k$.

$\therefore x=3-5k-8+2k$

$\quad=-5-3k$

The solution set is $\{(-5-3k,4-k,k)\,|\,k\in R\}$; therefore, the system is dependent.

2. The augmented matrix is $\begin{bmatrix}1&-2&3&4\\3&1&3&5\\2&4&1&2\end{bmatrix}$. ① ② ③

$③-2\times①$: $\begin{bmatrix}1&-2&3&4\\3&1&3&5\\0&8&-5&-6\end{bmatrix}$

$②-3\times①$: $\begin{bmatrix}1&-2&3&4\\0&7&-6&-7\\0&8&-5&-6\end{bmatrix}$

$③\times7-②\times8$: $\begin{bmatrix}1&-2&3&4\\0&7&-6&-7\\0&0&13&14\end{bmatrix}$

$②\times\frac{1}{7}$: $\begin{bmatrix}1&-2&3&4\\0&1&\frac{-6}{7}&-1\\0&0&1&\frac{14}{13}\end{bmatrix}$

$③\times\frac{1}{13}$:

From ③ you can state $z=\frac{14}{13}$.

From ② you can state $y-\frac{6}{7}z=-1$.

$\therefore y=-1+\left(\frac{6}{7}\right)\left(\frac{14}{13}\right)=\frac{-7}{91}=\frac{-1}{13}$

From ① you can state $x-2y+3z=4$.

$$\therefore x=4+2\left(\frac{-1}{13}\right)-3\left(\frac{14}{13}\right)=\frac{8}{13}$$

The solution set is $\left\{\left(\frac{8}{13},\frac{-1}{13},\frac{14}{13}\right)\right\}$; therefore, the system is independent.

3. $\begin{bmatrix}1 & \frac{8}{5}\\ 0 & 1\end{bmatrix}$ ①
②

$\frac{8}{5}\times$② : $\begin{bmatrix}1 & \frac{8}{5}\\ 0 & \frac{8}{5}\end{bmatrix}$

① − ② : $\begin{bmatrix}1 & 0\\ 0 & \frac{8}{5}\end{bmatrix}$

$\frac{5}{8}\times$② : $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$5\times$② + ① : $\begin{bmatrix}1 & 5\\ 0 & 1\end{bmatrix}$

$-17\times$② : $\begin{bmatrix}1 & 5\\ 0 & -17\end{bmatrix}$

Therefore, the two matrices are row equivalent.

Extra Help

1. a. The matrix of coefficients is $\begin{bmatrix}4 & -1\\ 2 & 3\end{bmatrix}$.

   b. The augmented matrix is $\begin{bmatrix}4 & -1 & 5\\ 2 & 3 & 6\end{bmatrix}$.

2. The following system can be obtained from the given matrix:

$$\left.\begin{aligned} x+4y-z&=2\\ -2x-y+4z&=6\\ 3x+5y+3z&=5\end{aligned}\right\}$$

3. $\begin{bmatrix}2 & -1 & 2\\ 1 & 5 & -3\\ 3 & 4 & 1\end{bmatrix}$ ①
②
③

   a. ① ↔ ③ : $\begin{bmatrix}3 & 4 & 1\\ 1 & 5 & -3\\ 2 & -1 & 2\end{bmatrix}$

   b. $-2\times$② + ③ : $\begin{bmatrix}3 & 4 & 1\\ 1 & 5 & -3\\ 0 & -11 & 8\end{bmatrix}$

c. $2\times(1)+(2)$: $\begin{bmatrix}3 & 4 & 1\\ 7 & 13 & -1\\ 0 & -11 & 8\end{bmatrix}$

4.

System of Equations	Matrix of Coefficients	Augmented Matrix
$x+y=8$ $2x-y=3$	$\begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix}$	$\begin{bmatrix}1 & 1 & 8\\ 2 & -1 & 3\end{bmatrix}$
$4x-5y=-3$ $x+6y=10$	$\begin{bmatrix}4 & -5\\ 1 & 6\end{bmatrix}$	$\begin{bmatrix}4 & -5 & -3\\ 1 & 6 & 10\end{bmatrix}$
$x+2y-z=2$ $2x-3y+z=5$ $3x+5y+6z=9$	$\begin{bmatrix}1 & 2 & -1\\ 2 & -3 & 1\\ 3 & 5 & 6\end{bmatrix}$	$\begin{bmatrix}1 & 2 & -1 & 2\\ 2 & -3 & 1 & 5\\ 3 & 5 & 6 & 9\end{bmatrix}$

5. $\begin{bmatrix}2 & -3 & 7\\ 5 & 0 & -4\end{bmatrix}$

a. $(1)\leftrightarrow(2)$: $\begin{bmatrix}5 & 0 & -4\\ 2 & -3 & 7\end{bmatrix}$

b. $2\times(1)+(2)$: $\begin{bmatrix}5 & 0 & -4\\ 12 & -3 & -1\end{bmatrix}$

c. $3\times(1)$: $\begin{bmatrix}15 & 0 & -12\\ 12 & -3 & -1\end{bmatrix}$

6. a. The matrix of coefficients is $\begin{bmatrix}2 & -1 & -3\\ 3 & -4 & 1\\ -1 & 3 & 2\end{bmatrix}$.

b. The augmented matrix is $\begin{bmatrix}2 & -1 & -3 & 7\\ 3 & -4 & 1 & 5\\ -1 & 3 & 2 & 10\end{bmatrix}$.

7. a. The following system can be obtained from the matrix:

$$\left.\begin{aligned}4x+2y+3z&=-4\\ 2x-3y-z&=0\end{aligned}\right\}$$

b. The following system can be obtained from the matrix:

$$\left.\begin{aligned}5x+3y+z&=8\\ x+y+2z&=-5\\ 2x-y+6z&=4\end{aligned}\right\}$$

8. $\begin{bmatrix} 2 & -1 & 6 \\ 1 & 1 & 2 \\ 5 & 3 & 1 \end{bmatrix} \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$

$(1) \leftrightarrow (3)$: $\begin{bmatrix} 5 & 3 & 1 \\ 1 & 1 & 2 \\ 2 & -1 & 6 \end{bmatrix}$

$2 \times (2)$: $\begin{bmatrix} 5 & 3 & 1 \\ 2 & 2 & 4 \\ 2 & -1 & 6 \end{bmatrix}$

$3 \times (2) + (3)$: $\begin{bmatrix} 5 & 3 & 1 \\ 2 & 2 & 4 \\ 8 & 5 & 18 \end{bmatrix}$

$(1) \leftrightarrow (3)$: $\begin{bmatrix} 8 & 5 & 18 \\ 2 & 2 & 4 \\ 5 & 3 & 1 \end{bmatrix}$

Extensions

1. $\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 2 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2+0 & 1+1 & 3+2 \\ 1-1 & -1+1 & 0+2 \end{bmatrix}$

$= \begin{bmatrix} 2 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix}$

2. You cannot find the sum of $\begin{bmatrix} 0 & 1 & 2 \\ 3 & -1 & 4 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 3 \\ 3 & 0 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ because the sum of matrices of different dimension is not defined.

3. If $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 0 \\ 2 & 3 & 6 \end{bmatrix}$, then $A^T = \begin{bmatrix} 3 & 4 & 2 \\ 2 & 5 & 3 \\ 1 & 0 & 6 \end{bmatrix}$.

4. $A \times B = \begin{bmatrix} 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} (1)(2)+(3)(1)+(-1)(3) \end{bmatrix}$

$= [2+3-3]$

$= [2]$

5. If $A=\begin{bmatrix} 1 & 2 & 0 & 3 \\ 4 & -1 & -2 & 6 \\ 0 & 3 & 1 & -4 \\ -4 & 5 & 6 & 0 \end{bmatrix}$, then $A^T=\begin{bmatrix} 1 & 4 & 0 & -4 \\ 2 & -1 & 3 & 5 \\ 0 & -2 & 1 & 6 \\ 3 & 6 & -4 & 0 \end{bmatrix}$.

6. $A\times B=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} u & v \\ w & x \end{bmatrix}=\begin{bmatrix} au+bw & av+bx \\ cu+dw & cv+dx \end{bmatrix}$

7. $A\times B=\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 1(6)+2(-1)+3(-1) & 1(-2)+2(1)+3(0) & 1(-3)+2(0)+3(1) \\ 1(6)+3(-1)+3(-1) & 1(-2)+3(1)+3(0) & 1(-3)+3(0)+3(1) \\ 1(6)+2(-1)+4(-1) & 1(-2)+2(1)+4(0) & 1(-3)+2(0)+4(1) \end{bmatrix}$

$=\begin{bmatrix} 6-5 & 2-2 & 3-3 \\ 6-6 & 3-2 & 3-3 \\ 6-6 & 2-2 & 4-3 \end{bmatrix}$

$=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. $A\times B=\begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}=\begin{bmatrix} -2(2)+1(4) & -2(3)+1(6) \\ -4(2)+2(4) & -4(3)+2(6) \end{bmatrix}$

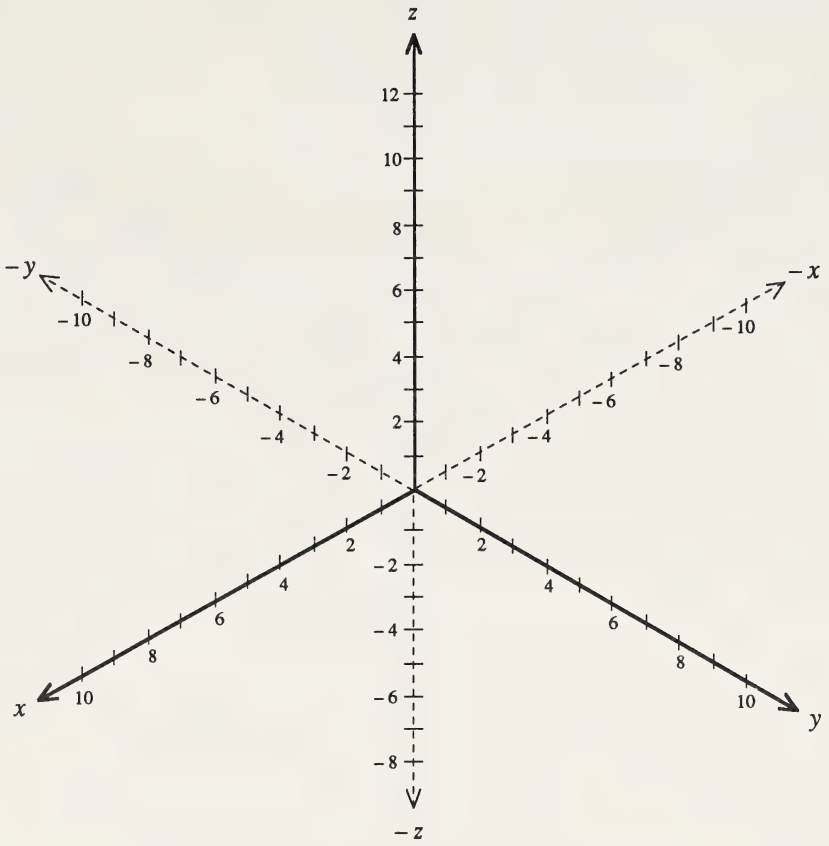
$=\begin{bmatrix} -4+4 & -6+6 \\ -8+8 & -12+12 \end{bmatrix}$

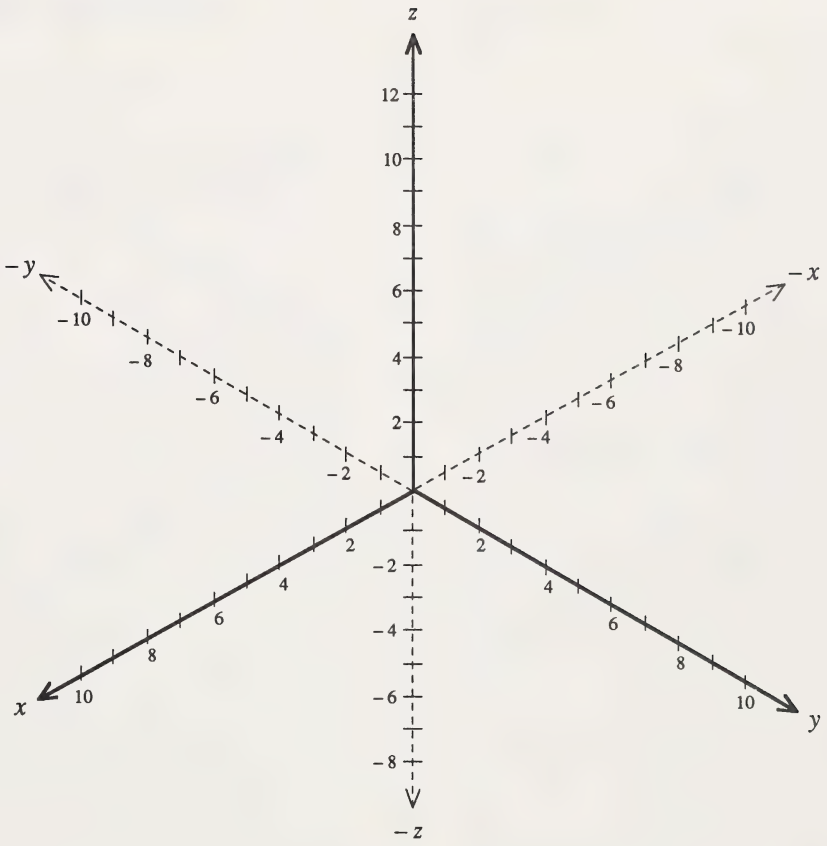
$=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Appendix B
Graphing Materials

Graph Paper

x-y-z Axes

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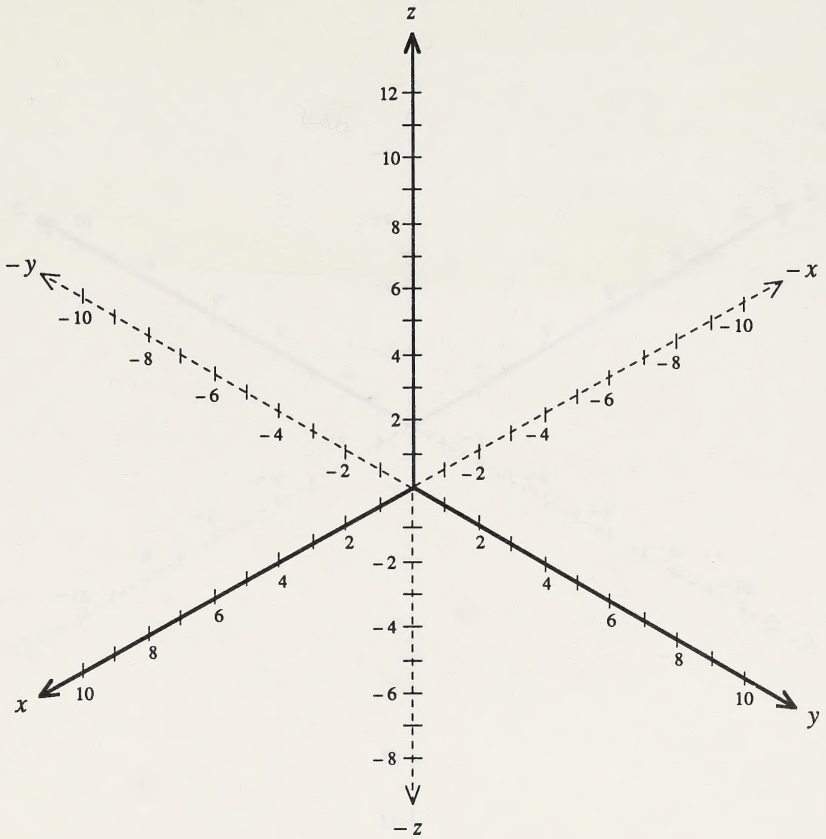
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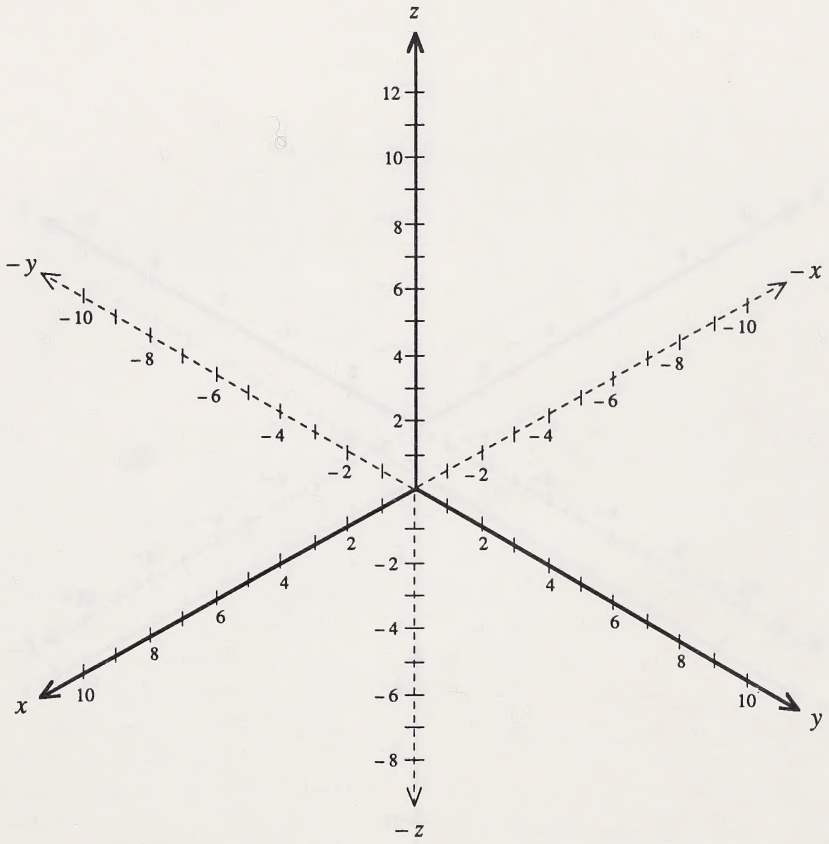
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